

EJERCICIOS DE INTEGRALES INDEFINIDAS

Ejercicio 1.-

Calcula las siguientes integrales:

a) $\int 7x^4$

b) $\int \frac{1}{x^2}$

c) $\int \sqrt{x}$

d) $\int \sqrt[3]{5x^2}$

e) $\int \frac{\sqrt[3]{x} + \sqrt{5x^3}}{3x}$

f) $\int \frac{\sqrt{5x^3}}{\sqrt[3]{3x}}$

a) $\int 7x^4 = 7 \frac{x^5}{5} + k = \frac{7x^5}{5} + k$

b) $\int \frac{1}{x^2} = \int x^{-2} = \frac{x^{-1}}{-1} + k = \frac{-1}{x} + k$

c) $\int \sqrt{x} = \int x^{1/2} = \frac{x^{3/2}}{3/2} + k = \frac{2\sqrt{x^3}}{3} + k$

d) $\int \sqrt[3]{5x^2} = \int \sqrt[3]{5} x^{2/3} = \sqrt[3]{5} \frac{x^{5/3}}{5/3} + k = \frac{3 \sqrt[3]{5x^5}}{5} + k$

e) $\int \frac{\sqrt[3]{x} + \sqrt{5x^3}}{3x} = \int \frac{x^{1/3}}{3x} + \int \frac{\sqrt{5}x^{3/2}}{3x} = \frac{1}{3} \int x^{-2/3} + \frac{\sqrt{5}}{3} \int x^{1/2} =$
 $= \frac{1}{3} \frac{x^{1/3}}{1/3} + \frac{\sqrt{5}}{3} \frac{x^{3/2}}{3/2} + k = \sqrt[3]{x} + \frac{2\sqrt{5}x^{3/2}}{9} + k$

f) $\int \frac{\sqrt{5x^3}}{\sqrt[3]{3x}} = \int \frac{\sqrt{5} \cdot x^{3/2}}{\sqrt[3]{3} \cdot x^{1/3}} = \frac{\sqrt{5}}{\sqrt[3]{3}} \int x^{7/6} = \frac{\sqrt{5}}{\sqrt[3]{3}} \frac{x^{13/6}}{13/6} + k = \frac{6\sqrt{5} \sqrt[6]{x^{13}}}{13\sqrt[3]{3}} + k$

Ejercicio 2.-

Calcula:

a) $\int \frac{x^4 - 5x^2 + 3x - 4}{x}$

b) $\int \frac{x^4 - 5x^2 + 3x - 4}{x + 1}$

c) $\int \frac{x^4 - 5x^2 + 3x - 4}{x^2 + 1}$

d) $\int \frac{x^3}{x - 2}$

$$a) \int \frac{x^4 - 5x^2 + 3x - 4}{x} = \int \left(x^3 - 5x + 3 - \frac{4}{x} \right) = \frac{x^4}{4} - \frac{5x^2}{2} + 3x - 4 \ln |x| + k$$

$$b) \int \frac{x^4 - 5x^2 + 3x - 4}{x + 1} = \int \left(x^3 - x^2 - 4x + 7 - \frac{11}{x + 1} \right) = \\ = \frac{x^4}{4} - \frac{x^3}{3} - 2x^2 + 7x - 11 \ln |x + 1| + k$$

$$c) \int \frac{x^4 - 5x^2 + 3x - 4}{x^2 + 1} = \int \left(x^2 - 6 + \frac{3x + 2}{x^2 + 1} \right) = \int \left(x^2 - 6 + \frac{3x}{x^2 + 1} + \frac{2}{x^2 + 1} \right) = \\ = \int x^2 - \int 6 + \frac{3}{2} \int \frac{2x}{x^2 + 1} + 2 \int \frac{1}{x^2 + 1} = \\ = \frac{x^3}{3} - 6x + \frac{3}{2} \ln (x^2 + 1) + 2 \operatorname{arc} \operatorname{tg} x + k$$

$$d) \int \frac{x^3}{x - 2} = \int \left(x^2 + 2x + 4 + \frac{8}{x - 2} \right) = \frac{x^3}{3} + x^2 + 4x + 8 \ln |x - 2| + k$$

Ejercicio 3.-

Calcula:

$$a) \int \cos^4 x \operatorname{sen} x \, dx$$

$$b) \int 2^{\operatorname{sen} x} \cos x \, dx$$

$$a) \int \cos^4 x \operatorname{sen} x \, dx = - \int \cos^4 x (-\operatorname{sen} x) \, dx = - \frac{\cos^5 x}{5} + k$$

$$b) \int 2^{\operatorname{sen} x} \cos x \, dx = \frac{1}{\ln 2} \int 2^{\operatorname{sen} x} \cos x \cdot \ln 2 \, dx = \frac{2^{\operatorname{sen} x}}{\ln 2} + k$$

Ejercicio 4.-

Calcula:

$$a) \int \cotg x \, dx$$

$$b) \int \frac{5x}{x^4 + 1} \, dx$$

$$a) \int \cotg x \, dx = \int \frac{\cos x}{\operatorname{sen} x} \, dx = \ln |\operatorname{sen} x| + k$$

$$b) \int \frac{5x}{x^4 + 1} \, dx = \frac{5}{2} \int \frac{2x}{1 + (x^2)^2} \, dx = \frac{5}{2} \operatorname{arc} \operatorname{tg} (x^2) + k$$

Ejercicio 5.-

Calcula: $\int \frac{1}{\sqrt[3]{x^2} - \sqrt{x}} dx$

Hacemos el cambio $x = t^6$, $dx = 6t^5 dt$:

$$\begin{aligned}\int \frac{1}{\sqrt[3]{x^2} - \sqrt{x}} dx &= \int \frac{1}{\sqrt[3]{t^{12}} - \sqrt{t^6}} 6t^5 dt = \int \frac{6t^5}{t^4 - t^3} dt = \int \frac{6t^2}{t-1} dt = 6 \int \frac{t^2}{t-1} dt = \\ &= 6 \int \left(t+1 + \frac{1}{t-1} \right) dt = 6 \int \left(\frac{t^2}{2} + t - \ln |t-1| \right) dt = k = \\ &= 6 \left(\frac{\sqrt[6]{x^2}}{2} + \sqrt[6]{x} - \ln |\sqrt[6]{x} - 1| \right) + k = 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6 \ln |\sqrt[6]{x} - 1| + k\end{aligned}$$

Ejercicio 6.-

Calcula: $\int \frac{x}{\sqrt{1-x^2}} dx$

Hacemos el cambio $\sqrt{1-x^2} = t \rightarrow 1-x^2 = t^2 \rightarrow x = \sqrt{1-t^2}$

$$dx = \frac{-t}{\sqrt{1-t^2}} dt$$

$$\int \frac{x}{\sqrt{1-x^2}} dx = \int \frac{\sqrt{1-t^2}}{t^2} \cdot \frac{-t}{\sqrt{1-t^2}} dt = \int -1 dt = -t + k = -\sqrt{1-x^2} + k$$

Ejercicio 7.-

Calcula: $\int x \operatorname{sen} x dx$

Llamamos $I = \int x \operatorname{sen} x dx$.

$$\left. \begin{aligned} u &= x, \quad du = dx \\ dv &= \operatorname{sen} x dx, \quad v = -\cos x \end{aligned} \right\} I = -x \cos x + \int \cos x dx = -x \cos x + \operatorname{sen} x + k$$

Ejercicio 8.-

Calcula: $\int x \operatorname{arc} \operatorname{tg} x dx$

Llamamos $I = \int x \operatorname{arc} \operatorname{tg} x dx$.

$$\left. \begin{aligned} u &= \operatorname{arc} \operatorname{tg} x, \quad du = \frac{1}{1+x^2} dx \\ dv &= x dx, \quad v = \frac{x^2}{2} \end{aligned} \right\}$$

$$\begin{aligned}I &= \frac{x^2}{2} \operatorname{arc} \operatorname{tg} x - \frac{1}{2} \int \left(\frac{x^2}{1+x^2} \right) dx = \frac{x^2}{2} \operatorname{arc} \operatorname{tg} x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx = \\ &= \frac{x^2}{2} \operatorname{arc} \operatorname{tg} x - \frac{1}{2} [x - \operatorname{arc} \operatorname{tg} x] + k = \frac{x^2}{2} \operatorname{arc} \operatorname{tg} x - \frac{1}{2} x + \frac{1}{2} \operatorname{arc} \operatorname{tg} x + k = \\ &= \frac{x^2+1}{2} \operatorname{arc} \operatorname{tg} x - \frac{1}{2} x + k\end{aligned}$$

Ejercicio 9.-

Calcula: $\int \frac{3x^2 - 5x + 1}{x - 4} dx$

$$\int \frac{3x^2 - 5x + 1}{x - 4} dx = \int \left(3x + 7 + \frac{29}{x - 4} \right) dx = \frac{3x^2}{2} + 7x + 29 \ln |x - 4| + k$$

Ejercicio 10.-

Calcula:

a) $\int \frac{5x - 3}{x^3 - x} dx$

b) $\int \frac{x^2 - 2x + 6}{(x - 1)^3} dx$

a) Descomponemos la fracción:

$$\frac{5x - 3}{x^3 - x} = \frac{5x - 3}{x(x - 1)(x + 1)} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{x + 1}$$

$$\frac{5x - 3}{x^3 - x} = \frac{A(x - 1)(x + 1) + Bx(x + 1) + Cx(x - 1)}{x(x - 1)(x + 1)}$$

$$5x - 3 = A(x - 1)(x + 1) + Bx(x + 1) + Cx(x - 1)$$

Hallamos A , B y C dando a x los valores 0, 1 y -1:

$$\left. \begin{array}{l} x = 0 \Rightarrow -3 = -A \Rightarrow A = 3 \\ x = 1 \Rightarrow 2 = 2B \Rightarrow B = 1 \\ x = -1 \Rightarrow -8 = 2C \Rightarrow C = -4 \end{array} \right\}$$

Así, tenemos que:

$$\int \frac{5x - 3}{x^3 - x} dx = \int \left(\frac{3}{x} + \frac{1}{x - 1} - \frac{4}{x + 1} \right) dx = 3 \ln |x| + \ln |x - 1| - 4 \ln |x + 1| + k$$

b) Descomponemos la fracción:

$$\frac{x^2 - 2x + 6}{(x - 1)^3} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{(x - 1)^3} = \frac{A(x - 1)^2 + B(x - 1) + C}{(x - 1)^3}$$

$$x^2 - 2x + 6 = A(x - 1)^2 + B(x - 1) + C$$

Dando a x los valores 1, 0 y 2, queda:

$$\left. \begin{array}{l} x = 1 \Rightarrow 5 = C \\ x = 0 \Rightarrow 6 = A - B + C \\ x = 2 \Rightarrow 6 = A + B + C \end{array} \right\} \begin{array}{l} A = 1 \\ B = 0 \\ C = 5 \end{array}$$

Por tanto:

$$\int \frac{x^2 - 2x + 6}{(x - 1)^3} dx = \int \left(\frac{1}{x - 1} + \frac{5}{(x - 1)^3} \right) dx = \ln |x - 1| - \frac{5}{2(x - 1)^2} + k$$

Ejercicio 11.-

Calcula:

$$a) \int \frac{x^3 + 22x^2 - 12x + 8}{x^4 - 4x^2} dx$$

$$b) \int \frac{x^3 - 4x^2 + 4x}{x^4 - 2x^3 - 4x^2 + 8x} dx$$

$$a) x^4 - 4x^2 = x^2(x^2 - 4) = x^2(x - 2)(x + 2)$$

Descomponemos la fracción:

$$\frac{x^3 + 22x^2 - 12x + 8}{x^2(x - 2)(x + 2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 2} + \frac{D}{x + 2}$$

$$\frac{x^3 + 22x^2 - 12x + 8}{x^2(x - 2)(x + 2)} =$$

$$= \frac{Ax(x - 2)(x + 2) + B(x - 2)(x + 2) + Cx^2(x + 2) + Dx^2(x - 2)}{x^2(x - 2)(x + 2)}$$

$$x^3 + 22x^2 - 12x + 8 = Ax(x - 2)(x + 2) + B(x - 2)(x + 2) + Cx^2(x + 2) + Dx^2(x - 2)$$

Hallamos A , B , C y D dando a x los valores 0, 2, -2 y 1:

$$\left. \begin{array}{l} x = 0 \Rightarrow 8 = -4B \Rightarrow B = -2 \\ x = 2 \Rightarrow 80 = 16C \Rightarrow C = 5 \\ x = -2 \Rightarrow 112 = -16D \Rightarrow D = -7 \\ x = 1 \Rightarrow 19 = -3A - 3B + 3C - D \Rightarrow -3A = -9 \Rightarrow A = 3 \end{array} \right\}$$

Por tanto:

$$\begin{aligned} \int \frac{x^3 + 22x^2 - 12x + 8}{x^4 - 4x^2} dx &= \int \left(\frac{3}{x} - \frac{2}{x^2} + \frac{5}{x - 2} - \frac{7}{x + 2} \right) dx = \\ &= 3 \ln|x| + \frac{2}{x} + 5 \ln|x - 2| - 7 \ln|x + 2| + k \end{aligned}$$

b) La fracción se puede simplificar:

$$\frac{x^3 - 4x^2 + 4x}{x^4 - 2x^3 - 4x^2 + 8x} = \frac{x(x - 2)^2}{x(x - 2)^2(x + 2)} = \frac{1}{x + 2}$$

$$\int \frac{x^3 - 4x^2 + 4x}{x^4 - 2x^3 - 4x^2 + 8x} dx = \int \frac{1}{x + 2} dx = \ln|x + 2| + k$$

Ejercicio 12.-

Calcula las siguientes integrales inmediatas:

$$a) \int (4x^2 - 5x + 7) dx \quad b) \int \frac{dx}{\sqrt[5]{x}} \quad c) \int \frac{1}{2x + 7} dx \quad d) \int (x - \operatorname{sen} x) dx$$

$$a) \int (4x^2 - 5x + 7) dx = \frac{4x^3}{3} - \frac{5x^2}{2} + 7x + k$$

$$b) \int \frac{dx}{\sqrt[5]{x}} = \int x^{-1/5} dx = \frac{x^{4/5}}{4/5} + k = \frac{5\sqrt[5]{x^4}}{4} + k$$

$$c) \int \frac{1}{2x+7} dx = \frac{1}{2} \ln |2x+7| + k$$

$$d) \int (x - \operatorname{sen} x) dx = \frac{x^2}{2} + \operatorname{cox} x + k$$

Ejercicio 13.-

Resuelve estas integrales:

$$a) \int (x^2 + 4x)(x^2 - 1) dx$$

$$b) \int (x-1)^3 dx$$

$$c) \int \sqrt{3x} dx$$

$$d) \int (\operatorname{sen} x + e^x) dx$$

$$a) \int (x^2 + 4x)(x^2 - 1) dx = \int (x^4 + 4x^3 - x^2 - 4x) dx = \frac{x^5}{5} + x^4 - \frac{x^3}{3} - 2x^2 + k$$

$$b) \int (x-1)^3 dx = \frac{(x-1)^4}{4} + k$$

$$c) \int \sqrt{3x} dx = \int \sqrt{3} x^{1/2} dx = \sqrt{3} \frac{x^{3/2}}{3/2} + k = \frac{2\sqrt{3}x^{3/2}}{3} + k$$

$$d) \int (\operatorname{sen} x + e^x) dx = -\operatorname{cos} x + e^x + k$$

Ejercicio 14.-

Calcula las integrales siguientes:

$$a) \int \sqrt[3]{\frac{x}{2}} dx \quad b) \int \operatorname{sen}(x-4) dx \quad c) \int \frac{7}{\cos^2 x} dx \quad d) \int (e^x + 3e^{-x}) dx$$

$$a) \int \sqrt[3]{\frac{x}{2}} dx = \frac{1}{\sqrt[3]{2}} \int x^{1/3} dx = \frac{1}{\sqrt[3]{2}} \frac{x^{4/3}}{4/3} + k = \frac{3}{4} \sqrt[3]{\frac{x^4}{2}} + k$$

$$b) \int \operatorname{sen}(x-4) dx = -\operatorname{cos}(x-4) + k$$

$$c) \int \frac{7}{\cos^2 x} dx = 7 \operatorname{tg} x + k$$

$$d) \int (e^x + 3e^{-x}) dx = e^x - 3e^{-x} + k$$

Ejercicio 15.-

Halla estas integrales:

$$a) \int \frac{2}{x} dx \quad b) \int \frac{dx}{x-1} \quad c) \int \frac{x + \sqrt{x}}{x^2} dx \quad d) \int \frac{3}{1+x^2} dx$$

$$a) \int \frac{2}{x} dx = 2 \ln |x| + k$$

$$b) \int \frac{dx}{x-1} = \ln|x-1| + k$$

$$c) \int \frac{x + \sqrt{x}}{x^2} dx = \int \left(\frac{1}{x} + x^{-3/2} \right) dx = \ln|x| - \frac{2}{\sqrt{x}} + k$$

$$d) \int \frac{3}{1+x^2} dx = 3 \operatorname{arc} \operatorname{tg} x + k$$

Ejercicio 16.-

Resuelve las siguientes integrales:

$$a) \int \frac{dx}{x-4} \quad b) \int \frac{dx}{(x-4)^2} \quad c) \int (x-4)^2 dx \quad d) \int \frac{dx}{(x-4)^3}$$

$$a) \int \frac{dx}{x-4} = \ln|x-4| + k$$

$$b) \int \frac{dx}{(x-4)^2} = \frac{-1}{(x-4)} + k$$

$$c) \int (x-4)^2 dx = \frac{(x-4)^3}{3} + k$$

$$d) \int \frac{dx}{(x-4)^3} = \int (x-4)^{-3} dx = \frac{(x-4)^{-2}}{-2} + k = \frac{-1}{2(x-4)^2} + k$$

Ejercicio 17.-

Halla las siguientes integrales del tipo exponencial:

$$a) \int e^{x-4} dx \quad b) \int e^{-2x+9} dx \quad c) \int e^{5x} dx \quad d) \int (3^x - x^3) dx$$

$$a) \int e^{x-4} dx = e^{x-4} + k$$

$$b) \int e^{-2x+9} dx = \frac{-1}{2} \int -2e^{-2x+9} dx = \frac{-1}{2} e^{-2x+9} + k$$

$$c) \int e^{5x} dx = \frac{1}{5} \int 5e^{5x} dx = \frac{1}{5} e^{5x} + k$$

$$d) \int (3^x - x^3) dx = \frac{3^x}{\ln 3} - \frac{x^4}{4} + k$$

Ejercicio 18.-

Resuelve las siguientes integrales del tipo arco tangente:

$$a) \int \frac{dx}{4+x^2} \quad b) \int \frac{4 dx}{3+x^2} \quad c) \int \frac{5 dx}{4x^2+1} \quad d) \int \frac{2 dx}{1+9x^2}$$

$$a) \int \frac{dx}{4+x^2} = \int \frac{1/4}{1+(x/2)^2} dx = \frac{1}{2} \int \frac{1/2}{1+(x/2)^2} dx = \frac{1}{2} \operatorname{arc} \operatorname{tg} \left(\frac{x}{2} \right) + k$$

$$b) \int \frac{4 \, dx}{3 + x^2} = \int \frac{4/3}{1 + (x/\sqrt{3})^2} \, dx = \frac{4\sqrt{3}}{3} \int \frac{1/\sqrt{3}}{1 + (x/\sqrt{3})^2} \, dx = \frac{4\sqrt{3}}{3} \operatorname{arc\,tg} \left(\frac{x}{\sqrt{3}} \right) + k$$

$$c) \int \frac{5 \, dx}{4x^2 + 1} = \frac{5}{2} \int \frac{2 \, dx}{(2x)^2 + 1} = \frac{5}{2} \operatorname{arc\,tg} (2x) + k$$

$$d) \int \frac{2 \, dx}{1 + 9x^2} = \frac{2}{3} \int \frac{3 \, dx}{1 + (3x)^2} = \frac{2}{3} \operatorname{arc\,tg} (3x) + k$$

Ejercicio 19.-

Expresa las siguientes integrales de la forma:

$$\frac{\text{dividendo}}{\text{divisor}} = \text{cociente} + \frac{\text{resto}}{\text{divisor}}$$

y resuélvelas:

$$a) \int \frac{x^2 - 5x + 4}{x + 1} \, dx \quad b) \int \frac{x^2 + 2x + 4}{x + 1} \, dx \quad c) \int \frac{x^3 - 3x^2 + x - 1}{x - 2} \, dx$$

$$a) \int \frac{x^2 - 5x + 4}{x + 1} \, dx = \int \left(x - 6 + \frac{10}{x + 1} \right) \, dx = \frac{x^2}{2} - 6x + 10 \ln |x + 1| + k$$

$$b) \int \frac{x^2 + 2x + 4}{x + 1} \, dx = \int \left(x + 1 + \frac{3}{x + 1} \right) \, dx = \frac{x^2}{2} + x + 3 \ln |x + 1| + k$$

$$c) \int \frac{x^3 - 3x^2 + x - 1}{x - 2} \, dx = \int \left(x^2 - x - 1 - \frac{3}{x - 2} \right) \, dx = \\ = \frac{x^3}{3} - \frac{x^2}{2} - x - 3 \ln |x - 2| + k$$

Ejercicio 20.-

Halla estas integrales sabiendo que son del tipo arco seno:

$$a) \int \frac{dx}{\sqrt{1 - 4x^2}} \quad b) \int \frac{dx}{\sqrt{4 - x^2}} \quad c) \int \frac{e^x}{\sqrt{1 - e^{2x}}} \, dx \quad d) \int \frac{dx}{x \sqrt{1 - (\ln x)^2}}$$

$$a) \int \frac{dx}{\sqrt{1 - 4x^2}} = \frac{1}{2} \int \frac{2 \, dx}{\sqrt{1 - (2x)^2}} = \frac{1}{2} \operatorname{arc\,sen} (2x) + k$$

$$b) \int \frac{dx}{\sqrt{4 - x^2}} = \int \frac{1/2 \, dx}{\sqrt{1 - (x/2)^2}} = \operatorname{arc\,sen} \left(\frac{x}{2} \right) + k$$

$$c) \int \frac{e^x}{\sqrt{1 - e^{2x}}} \, dx = \int \frac{e^x}{\sqrt{1 - (e^x)^2}} \, dx = \operatorname{arc\,sen} (e^x) + k$$

$$d) \int \frac{dx}{x \sqrt{1 - (\ln x)^2}} = \int \frac{1/x \, dx}{\sqrt{1 - (\ln x)^2}} = \operatorname{arc\,sen} (\ln |x|) + k$$

Ejercicio 21.-

Resuelve las integrales siguientes,

a) $\int \cos x \operatorname{sen}^3 x \, dx$ b) $\int 2x e^{x^2} \, dx$ c) $\int \frac{x \, dx}{(x^2 + 3)^5}$ d) $\int \frac{1}{x} \ln^3 x \, dx$

a) $\int \cos x \operatorname{sen}^3 x \, dx = \frac{\operatorname{sen}^4 x}{4} + k$

b) $\int 2x e^{x^2} \, dx = e^{x^2} + k$

c) $\int \frac{x \, dx}{(x^2 + 3)^5} = \frac{1}{2} \int 2x(x^2 + 3)^{-5} \, dx = \frac{1}{2} \frac{(x^2 + 3)^{-4}}{-4} + k = \frac{-1}{8(x^2 + 3)^4} + k$

d) $\int \frac{1}{x} \ln^3 x \, dx = \frac{\ln^4 |x|}{4} + k$

Ejercicio 22.-

Resuelve las siguientes integrales:

a) $\int x^4 e^{x^5} \, dx$ b) $\int x \operatorname{sen} x^2 \, dx$ c) $\int \frac{dx}{\sqrt{9 - x^2}}$ d) $\int \frac{x \, dx}{\sqrt{x^2 + 5}}$

a) $\int x^4 e^{x^5} \, dx = \frac{1}{5} \int 5x^4 e^{x^5} \, dx = \frac{1}{5} e^{x^5} + k$

b) $\int x \operatorname{sen} x^2 \, dx = \frac{1}{2} \int 2x \operatorname{sen} x^2 \, dx = \frac{-1}{2} \cos x^2 + k$

c) $\int \frac{dx}{\sqrt{9 - x^2}} = \int \frac{1/3 \, dx}{\sqrt{1 - (x/3)^2}} = \operatorname{arc} \operatorname{sen} \left(\frac{x}{3} \right) + k$

d) $\int \frac{x \, dx}{\sqrt{x^2 + 5}} = \sqrt{x^2 + 5} + k$

Ejercicio 23.-

Resuelve las siguientes integrales:

a) $\int \sqrt{x^2 - 2x} (x - 1) \, dx$ b) $\int \operatorname{tg} x \sec^2 x \, dx$

c) $\int \frac{(1 + \ln x)^2}{x} \, dx$ d) $\int \sqrt{(1 + \cos x)^3} \operatorname{sen} x \, dx$

a) $\int \sqrt{x^2 - 2x} (x - 1) \, dx = \frac{1}{2} \int \sqrt{x^2 - 2x} (2x - 2) \, dx = \frac{1}{2} \int (x^2 - 2x)^{1/2} (2x - 2) \, dx =$
 $= \frac{1}{2} \frac{(x^2 - 2x)^{3/2}}{3/2} + k = \frac{\sqrt{(x^2 - 2x)^3}}{3} + k$

b) $\int \operatorname{tg} x \sec^2 x \, dx = \frac{\operatorname{tg}^2 x}{2} + k$

c) $\int \frac{(1 + \ln x)^2}{x} \, dx = \int (1 + \ln x)^2 \cdot \frac{1}{x} \, dx = \frac{(1 + \ln |x|)^3}{3} + k$

d) $\int \sqrt{(1 + \cos x)^3} \operatorname{sen} x \, dx = - \int (1 + \cos x)^{3/2} (-\operatorname{sen} x) \, dx = - \frac{(1 + \cos x)^{5/2}}{5/2} + k =$
 $= \frac{-2\sqrt{(1 + \cos x)^5}}{5} + k$

Ejercicio 24.-

Aplica la integración por partes para resolver las siguientes integrales:

a) $\int x \ln x \, dx$ b) $\int e^x \cos x \, dx$ c) $\int x^2 \operatorname{sen} x \, dx$ d) $\int x^2 e^{2x} \, dx$

e) $\int \cos(\ln x) \, dx$ f) $\int x^2 \ln x \, dx$ g) $\int \operatorname{arc} \operatorname{tg} x \, dx$ h) $\int (x+1)^2 e^x \, dx$

a) $\int x \ln x \, dx$

$$\begin{cases} u = \ln x \rightarrow du = \frac{1}{x} dx \\ dv = x \, dx \rightarrow v = \frac{x^2}{2} \end{cases}$$

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x}{2} \, dx = \frac{x^2}{2} \ln |x| - \frac{x^2}{4} + k$$

b) $\int e^x \cos x \, dx$

$$\begin{cases} u = e^x \rightarrow du = e^x dx \\ dv = \cos x \, dx \rightarrow v = \operatorname{sen} x \end{cases}$$

$$\int e^x \cos x \, dx = e^x \operatorname{sen} x - \underbrace{\int e^x \operatorname{sen} x \, dx}_{I_1}$$

$$\begin{cases} u_1 = e^x \rightarrow du_1 = e^x dx \\ dv_1 = \operatorname{sen} x \, dx \rightarrow v_1 = -\cos x \end{cases}$$

$$I_1 = -e^x \cos x + \int e^x \cos x \, dx$$

Por tanto:

$$\int e^x \cos x \, dx = e^x \operatorname{sen} x + e^x \cos x - \int e^x \cos x \, dx$$

$$2 \int e^x \cos x \, dx = e^x \operatorname{sen} x + e^x \cos x$$

$$\int e^x \cos x \, dx = \frac{e^x \operatorname{sen} x + e^x \cos x}{2} + k$$

$$c) \int x^2 \operatorname{sen} x \, dx$$

$$\begin{cases} u = x^2 & \rightarrow & du = 2x \, dx \\ dv = \operatorname{sen} x \, dx & \rightarrow & v = -\cos x \end{cases}$$

$$\int x^2 \operatorname{sen} x \, dx = -x^2 \cos x + \int 2x \cos x \, dx = -x^2 \cos x + 2 \underbrace{\int x \cos x \, dx}_{I_1}$$

$$\begin{cases} u_1 = x & \rightarrow & du_1 = dx \\ dv_1 = \cos x \, dx & \rightarrow & v_1 = \operatorname{sen} x \end{cases}$$

$$I_1 = x \operatorname{sen} x - \int \operatorname{sen} x \, dx = x \operatorname{sen} x + \cos x$$

Por tanto:

$$\int x^2 \operatorname{sen} x \, dx = -x^2 \cos x + 2x \operatorname{sen} x + 2 \cos x + k$$

$$d) \int x^2 e^{2x} \, dx$$

$$\begin{cases} u = x^2 & \rightarrow & du = 2x \, dx \\ dv = e^{2x} \, dx & \rightarrow & v = \frac{1}{2} e^{2x} \end{cases}$$

$$\int x^2 e^{2x} \, dx = \frac{x^2}{2} e^{2x} - \underbrace{\int x e^{2x} \, dx}_{I_1}$$

$$\begin{cases} u_1 = x & \rightarrow & du_1 = dx \\ dv_1 = e^{2x} \, dx & \rightarrow & v_1 = \frac{1}{2} e^{2x} \end{cases}$$

$$I_1 = \frac{x}{2} e^{2x} - \int \frac{1}{2} e^{2x} \, dx = \frac{x}{2} e^{2x} - \frac{1}{4} e^{2x}$$

Por tanto: $\int x^2 e^{2x} \, dx = \frac{x^2}{2} e^{2x} - \frac{x}{2} e^{2x} + \frac{1}{4} e^{2x} + k = \left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4} \right) e^{2x} + k$

$$e) \int \cos(\ln x) dx$$

$$\begin{cases} u = \cos(\ln x) \rightarrow du = -\operatorname{sen}(\ln x) \cdot \frac{1}{x} dx \\ dv = dx \rightarrow v = x \end{cases}$$

$$\int \cos(\ln x) dx = x \cos(\ln x) + \underbrace{\int \operatorname{sen}(\ln x) dx}_{I_1}$$

$$\begin{cases} u_1 = \operatorname{sen}(\ln x) \rightarrow du_1 = \cos(\ln x) \cdot \frac{1}{x} dx \\ dv_1 = dx \rightarrow v_1 = x \end{cases}$$

$$I_1 = x \operatorname{sen}(\ln x) - \int \cos(\ln x) dx$$

Por tanto:

$$\int \cos(\ln x) dx = x \cos(\ln x) + x \operatorname{sen}(\ln x) - \int \cos(\ln x) dx$$

$$2 \int \cos(\ln x) dx = x \cos(\ln x) + x \operatorname{sen}(\ln x)$$

$$\int \cos(\ln x) dx = \frac{x \cos(\ln x) + x \operatorname{sen}(\ln x)}{2} + k$$

$$f) \int x^2 \ln x dx$$

$$\begin{cases} u = \ln x \rightarrow du = \frac{1}{x} dx \\ dv = x^2 dx \rightarrow v = \frac{x^3}{3} \end{cases}$$

$$\int x^2 \ln x dx = \frac{x^3 \ln x}{3} - \int \frac{x^2}{3} dx = \frac{x^3 \ln x}{3} - \frac{x^3}{9} + k$$

$$g) \int \arctan x \, dx$$

$$\begin{cases} u = \arctan x & \rightarrow & du = \frac{1}{1+x^2} \, dx \\ dv = dx & \rightarrow & v = x \end{cases}$$

$$\begin{aligned} \int \arctan x &= x \arctan x - \int \frac{1}{1+x^2} \, dx = x \arctan x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx = \\ &= x \arctan x - \frac{1}{2} \ln(1+x^2) + k \end{aligned}$$

$$h) \int (x+1)^2 e^x \, dx$$

$$\begin{cases} u = (x+1)^2 & \rightarrow & du = 2(x+1) \, dx \\ dv = e^x \, dx & \rightarrow & v = e^x \end{cases}$$

$$\int (x+1)^2 e^x \, dx = (x+1)^2 e^x - \underbrace{2 \int (x+1) e^x \, dx}_{I_1}$$

$$\begin{cases} u_1 = (x+1) & \rightarrow & du_1 = dx \\ dv_1 = e^x \, dx & \rightarrow & v_1 = e^x \end{cases}$$

$$I_1 = (x+1) e^x - \int e^x \, dx = (x+1) e^x - e^x = (x+1-1) e^x = x e^x$$

Por tanto:

$$\begin{aligned} \int (x+1)^2 e^x \, dx &= (x+1)^2 e^x - 2x e^x + k = \\ &= (x^2 + 2x + 1 - 2x) e^x + k = (x^2 + 1) e^x + k \end{aligned}$$

Ejercicio 25.-

Determina el valor de las integrales que se proponen a continuación:

a) $\int x \cdot 2^{-x} dx$ b) $\int \arccos x dx$ c) $\int x \cos 3x dx$ d) $\int x^5 e^{-x^3} dx$

a) $\int x \cdot 2^{-x} dx$

$$\begin{cases} u = x \rightarrow du = dx \\ dv = 2^{-x} dx \rightarrow v = \frac{-2^{-x}}{\ln 2} \end{cases}$$

$$\begin{aligned} \int x 2^{-x} dx &= \frac{-x \cdot 2^{-x}}{\ln 2} + \int \frac{2^{-x}}{\ln 2} dx = \frac{-x \cdot 2^{-x}}{\ln 2} + \frac{1}{\ln 2} \int 2^{-x} dx = \\ &= \frac{-x \cdot 2^{-x}}{\ln 2} - \frac{2^{-x}}{(\ln 2)^2} + k \end{aligned}$$

b) $\int \arccos x dx$

$$\begin{cases} u = \arccos x \rightarrow du = \frac{-1}{\sqrt{1-x^2}} dx \\ dv = dx \rightarrow v = x \end{cases}$$

$$\int \arccos x dx = x \arccos x - \int \frac{-x}{\sqrt{1-x^2}} dx = x \arccos x - \sqrt{1-x^2} + k$$

c) $\int x \cos 3x dx$

$$\begin{cases} u = x \rightarrow du = dx \\ dv = \cos 3x dx \rightarrow v = \frac{1}{3} \sin 3x \end{cases}$$

$$\int x \cos 3x dx = \frac{x}{3} \sin 3x - \frac{1}{3} \int \sin 3x dx = \frac{x}{3} \sin 3x + \frac{1}{9} \cos 3x + k$$

d) $\int x^5 e^{-x^3} dx = \underbrace{\int x^3}_{u} \cdot \underbrace{x^2 e^{-x^3} dx}_{dv}$

$$\begin{cases} u = x^3 \rightarrow du = 3x^2 dx \\ dv = x^2 e^{-x^3} dx \rightarrow v = \frac{-1}{3} e^{-x^3} \end{cases}$$

$$\begin{aligned} \int x^5 e^{-x^3} dx &= \frac{-x^3}{3} e^{-x^3} + \int x^2 e^{-x^3} dx = \frac{-x^3}{3} e^{-x^3} - \frac{1}{3} e^{-x^3} + k = \\ &= \frac{(-x^3 - 1)}{3} e^{-x^3} + k \end{aligned}$$

Ejercicio 26.-

Resuelve las siguientes integrales:

a) $\int \frac{2x-4}{(x-1)^2 (x+3)} dx$

b) $\int \frac{2x+3}{(x-2)(x+5)} dx$

c) $\int \frac{1}{(x-1)(x+3)^2} dx$

d) $\int \frac{3x-2}{x^2-4} dx$

a) $\int \frac{2x-4}{(x-1)^2 (x+3)} dx$

Descomponemos en fracciones simples:

$$\frac{2x-4}{(x-1)^2 (x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$$

$$\frac{2x-4}{(x-1)^2 (x+3)} = \frac{A(x-1)(x+3) + B(x+3) + C(x-1)^2}{(x-1)^2 (x+3)}$$

$$2x-4 = A(x-1)(x+3) + B(x+3) + C(x-1)^2$$

Hallamos A , B y C :

$$\left. \begin{array}{lll} x=1 & \rightarrow & -2 = 4B \quad \rightarrow \quad B = -1/2 \\ x=-3 & \rightarrow & -10 = 16C \quad \rightarrow \quad C = -5/8 \\ x=0 & \rightarrow & -4 = -3A + 3B + C \quad \rightarrow \quad A = 5/8 \end{array} \right\}$$

Por tanto:

$$\begin{aligned} \int \frac{2x-4}{(x-1)^2 (x+3)} dx &= \int \frac{5/8}{x-1} dx + \int \frac{-1/2}{(x-1)^2} dx + \int \frac{-5/8}{x+3} dx = \\ &= \frac{5}{8} \ln|x-1| + \frac{1}{2} \cdot \frac{1}{(x-1)} - \frac{5}{8} \ln|x+3| + k = \frac{5}{8} \ln \left| \frac{x-1}{x+3} \right| + \frac{1}{2x-2} + k \end{aligned}$$

b) $\int \frac{2x+3}{(x-2)(x+5)} dx$

Descomponemos en fracciones simples:

$$\frac{2x+3}{(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5} = \frac{A(x+5) + B(x-2)}{(x-2)(x+5)}$$

$$2x+3 = A(x+5) + B(x-2)$$

Hallamos A y B :

$$\left. \begin{array}{lll} x=2 & \rightarrow & 7 = 7A \quad \rightarrow \quad A = 1 \\ x=-5 & \rightarrow & -7 = -7B \quad \rightarrow \quad B = 1 \end{array} \right\}$$

Por tanto:

$$\begin{aligned} \int \frac{2x+3}{(x-2)(x+5)} dx &= \int \frac{1}{x-2} dx + \int \frac{1}{x+5} dx = \\ &= \ln|x-2| + \ln|x+5| + k = \ln|(x-2)(x+5)| + k \end{aligned}$$

$$c) \int \frac{1}{(x-1)(x+3)^2} dx$$

Descomponemos en fracciones simples:

$$\frac{1}{(x-1)(x+3)^2} = \frac{A}{x-1} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$$

$$\frac{1}{(x-1)(x+3)^2} = \frac{A(x+3)^2 + B(x-1)(x+3) + C(x-1)}{(x-1)(x+3)^2}$$

$$1 = A(x+3)^2 + B(x-1)(x+3) + C(x-1)$$

Hallamos A , B y C :

$$\left. \begin{array}{lll} x=1 & \rightarrow & 1=16A \quad \rightarrow \quad A=1/16 \\ x=-3 & \rightarrow & 1=-4C \quad \rightarrow \quad C=-1/4 \\ x=0 & \rightarrow & 1=9A-3B-C \quad \rightarrow \quad B=-1/16 \end{array} \right\}$$

Por tanto:

$$\begin{aligned} \int \frac{1}{(x-1)(x+3)^2} dx &= \int \frac{1/16}{x-1} dx + \int \frac{-1/16}{x+3} dx + \int \frac{-1/4}{(x+3)^2} dx = \\ &= \frac{1}{16} \ln|x-1| - \frac{1}{16} \ln|x+3| + \frac{1}{4} \cdot \frac{1}{(x+3)} + k = \\ &= \frac{1}{16} \ln \left| \frac{x-1}{x+3} \right| + \frac{1}{4(x+3)} + k \end{aligned}$$

$$d) \int \frac{3x-2}{x^2-4} dx = \int \frac{3x-2}{(x-2)(x+2)} dx$$

Descomponemos en fracciones simples:

$$\frac{3x-2}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} = \frac{A(x+2) + B(x-2)}{(x-2)(x+2)}$$

$$3x-2 = A(x+2) + B(x-2)$$

Hallamos A y B :

$$\left. \begin{array}{lll} x=2 & \rightarrow & 4=4A \quad \rightarrow \quad A=1 \\ x=-2 & \rightarrow & -8=-4B \quad \rightarrow \quad B=2 \end{array} \right\}$$

Por tanto:

$$\begin{aligned} \int \frac{3x-2}{x^2-4} dx &= \int \frac{1}{x-2} dx + \int \frac{2}{x+2} dx = \\ &= \ln|x-2| + 2 \ln|x+2| + k = \ln[|x-2|(x+2)^2] + k \end{aligned}$$

Ejercicio 27.-

Resuelve las integrales:

a) $\int \frac{\ln x}{x} dx$

b) $\int \frac{1 - \operatorname{sen} x}{x + \cos x} dx$

c) $\int \frac{1}{x \ln x} dx$

d) $\int \frac{1 + e^x}{e^x + x} dx$

e) $\int \frac{\operatorname{sen}(1/x)}{x^2} dx$

f) $\int \frac{2x-3}{x+2} dx$

g) $\int \frac{\operatorname{arc} \operatorname{tg} x}{1+x^2} dx$

h) $\int \frac{\operatorname{sen} x}{\cos^4 x} dx$

a) $\int \frac{\ln x}{x} dx = \int \frac{1}{x} \ln x dx = \frac{\ln^2 |x|}{2} + k$

b) $\int \frac{1 - \operatorname{sen} x}{x + \cos x} dx = \ln |x + \cos x| + k$

c) $\int \frac{1}{x \ln x} dx = \int \frac{1/x}{\ln x} dx = \ln |\ln |x|| + k$

d) $\int \frac{1 + e^x}{e^x + x} dx = \ln |e^x + x| + k$

e) $\int \frac{\operatorname{sen}(1/x)}{x^2} dx = - \int \frac{-1}{x^2} \operatorname{sen}\left(\frac{1}{x}\right) dx = \cos\left(\frac{1}{x}\right) + k$

f) $\int \frac{2x-3}{x+2} dx = \int \left(2 - \frac{7}{x+2}\right) dx = 2x - 7 \ln |x+2| + k$

g) $\int \frac{\operatorname{arc} \operatorname{tg} x}{1+x^2} dx = \int \frac{1}{1+x^2} \operatorname{arc} \operatorname{tg} x dx = \frac{\operatorname{arc} \operatorname{tg}^2 x}{2} + k$

h) $\int \frac{\operatorname{sen} x}{\cos^4 x} dx = - \int (-\operatorname{sen} x) (\cos x)^{-4} dx = \frac{-(\cos x)^{-3}}{-3} + k = \frac{1}{3 \cos^3 x} + k$

Ejercicio 28.-

Resuelve por sustitución:

a) $\int x\sqrt{x+1} \, dx$

b) $\int \frac{dx}{x - \sqrt[4]{x}}$

c) $\int \frac{x}{\sqrt{x+1}} \, dx$

d) $\int \frac{1}{x\sqrt{x+1}} \, dx$

e) $\int \frac{1}{x + \sqrt{x}} \, dx$

f) $\int \frac{\sqrt{x}}{1+x} \, dx$

• a) Haz $x+1 = t^2$. b) Haz $x = t^4$.

a) $\int x\sqrt{x+1} \, dx$

Cambio: $x+1 = t^2 \rightarrow dx = 2t \, dt$

$$\begin{aligned} \int x\sqrt{x+1} \, dx &= \int (t^2 - 1)t \cdot 2t \, dt = \int (2t^4 - 2t^2) \, dt = \frac{2t^5}{5} - \frac{2t^3}{3} + k = \\ &= \frac{2\sqrt{(x+1)^5}}{5} - \frac{2\sqrt{(x+1)^3}}{3} + k \end{aligned}$$

b) $\int \frac{dx}{x - \sqrt[4]{x}}$

Cambio: $x = t^4 \rightarrow dx = 4t^3 \, dt$

$$\begin{aligned} \int \frac{dx}{x - \sqrt[4]{x}} &= \int \frac{4t^3 \, dt}{t^4 - t} = \int \frac{4t^2 \, dt}{t^3 - 1} = \frac{4}{3} \int \frac{3t^2 \, dt}{t^3 - 1} = \frac{4}{3} \ln|t^3 - 1| + k = \\ &= \frac{4}{3} \ln|\sqrt[4]{x^3} - 1| + k \end{aligned}$$

c) $\int \frac{x}{\sqrt{x+1}} \, dx$

Cambio: $x+1 = t^2 \rightarrow dx = 2t \, dt$

$$\begin{aligned} \int \frac{x}{\sqrt{x+1}} \, dx &= \int \frac{(t^2 - 1)}{t} \cdot 2t \, dt = \int (2t^2 - 2) \, dt = \frac{2t^3}{3} - 2t + k = \\ &= \frac{2\sqrt{(x+1)^3}}{3} - 2\sqrt{x+1} + k \end{aligned}$$

d) $\int \frac{1}{x\sqrt{x+1}} \, dx$

Cambio: $x+1 = t^2 \rightarrow dx = 2t \, dt$

$$\int \frac{1}{x\sqrt{x+1}} \, dx = \int \frac{2t \, dt}{(t^2 - 1)t} = \int \frac{2 \, dt}{(t+1)(t-1)}$$

Descomponemos en fracciones simples:

$$\frac{2}{(t+1)(t-1)} = \frac{A}{t+1} + \frac{B}{t-1} = \frac{A(t-1) + B(t+1)}{(t+1)(t-1)}$$

$$2 = A(t-1) + B(t+1)$$

Hallamos A y B :

$$\left. \begin{array}{l} t = -1 \rightarrow 2 = -2A \rightarrow A = -1 \\ t = 1 \rightarrow 2 = 2B \rightarrow B = 1 \end{array} \right\}$$

Por tanto:

$$\begin{aligned} \int \frac{2 \, dt}{(t+1)(t-1)} &= \int \left(\frac{-1}{t+1} + \frac{1}{t-1} \right) dt = -\ln|t+1| + \ln|t-1| + k = \\ &= \ln \left| \frac{t-1}{t+1} \right| + k \end{aligned}$$

Así:

$$\int \frac{1}{x \sqrt{x+1}} dx = \ln \left| \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1} \right| + k$$

$$e) \int \frac{1}{x + \sqrt{x}} dx$$

$$\text{Cambio: } x = t^2 \rightarrow dx = 2t \, dt$$

$$\begin{aligned} \int \frac{1}{x + \sqrt{x}} dx &= \int \frac{2t \, dt}{t^2 + t} = \int \frac{2 \, dt}{t+1} = 2 \ln|t+1| + k = \\ &= 2 \ln(\sqrt{x} + 1) + k \end{aligned}$$

$$f) \int \frac{\sqrt{x}}{1+x} dx$$

$$\text{Cambio: } x = t^2 \rightarrow dx = 2t \, dt$$

$$\begin{aligned} \int \frac{\sqrt{x}}{1+x} dx &= \int \frac{t \cdot 2t \, dt}{1+t^2} = \int \frac{2t^2 \, dt}{1+t^2} = \int \left(2 - \frac{2}{1+t^2} \right) dt = \\ &= 2t - 2 \arctan t + k = 2\sqrt{x} - 2 \arctan \sqrt{x} + k \end{aligned}$$

Ejercicio 29.-

Resuelve, utilizando un cambio de variable, estas integrales:

$$a) \int \sqrt{9-4x^2} \, dx \quad b) \int \frac{dx}{e^{2x}-3e^x} \quad c) \int \frac{e^{3x}-e^x}{e^{2x}+1} dx \quad d) \int \frac{1}{1+\sqrt{x}} dx$$

• a) Haz $\sin t = 2x/3$.

$$a) \int \sqrt{9-4x^2} \, dx$$

$$\text{Cambio: } \sin t = \frac{2x}{3} \rightarrow x = \frac{3}{2} \sin t \rightarrow dx = \frac{3}{2} \cos t \, dt$$

$$\begin{aligned}
\int \sqrt{9-4x^2} \, dx &= \int \sqrt{9-4 \cdot \frac{9}{4} \sin^2 t} \cdot \frac{3}{2} \cos t \, dt = \int 3 \cos t \cdot \frac{3}{2} \cos t \, dt = \\
&= \frac{9}{2} \int \cos^2 t \, dt = \frac{9}{2} \int \left(\frac{1}{2} - \frac{\cos 2t}{2} \right) dt = \frac{9}{2} \left(\frac{1}{2} t + \frac{1}{4} \sin 2t \right) + k = \\
&= \frac{9}{4} t + \frac{9}{8} \sin 2t + k = \frac{9}{4} \arcsin \left(\frac{2x}{3} \right) + \frac{9}{8} \cdot 2 \sin t \cos t + k = \\
&= \frac{9}{4} \arcsin \left(\frac{2x}{3} \right) + \frac{9}{4} \cdot \frac{2x}{3} \sqrt{1 - \frac{4x^2}{9}} + k = \\
&= \frac{9}{4} \arcsin \left(\frac{2x}{3} \right) + \frac{x}{2} \cdot \sqrt{9-4x^2} + k
\end{aligned}$$

b) $\int \frac{dx}{e^{2x} - 3e^x}$

Cambio: $e^x = t \rightarrow x = \ln t \rightarrow dx = \frac{1}{t} dt$

$$\int \frac{dx}{e^{2x} - 3e^x} = \int \frac{1/t}{t^2 - 3t} dt = \int \frac{1}{t^3 - 3t^2} dt = \int \frac{1}{t^2(t-3)} dt$$

Descomponemos en fracciones simples:

$$\frac{1}{t^2(t-3)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t-3} = \frac{At(t-3) + B(t-3) + Ct^2}{t^2(t-3)}$$

$$1 = At(t-3) + B(t-3) + Ct^2$$

Hallamos A , B y C :

$$\left. \begin{aligned}
t=0 &\rightarrow 1 = -3B && \rightarrow B = -1/3 \\
t=3 &\rightarrow 1 = 9C && \rightarrow C = 1/9 \\
t=1 &\rightarrow 1 = -2A - 2B + C && \rightarrow A = -1/9
\end{aligned} \right\}$$

Así, tenemos que:

$$\begin{aligned}
\int \frac{1}{t^2(t-3)} dt &= \int \left(\frac{-1/9}{t} + \frac{-1/3}{t^2} + \frac{1/9}{t-3} \right) dt = \\
&= \frac{-1}{9} \ln|t| + \frac{1}{3t} + \frac{1}{9} \ln|t-3| + k
\end{aligned}$$

Por tanto:

$$\begin{aligned}
\int \frac{dx}{e^{2x} - 3e^x} &= \frac{-1}{9} \ln e^x + \frac{1}{3e^x} + \frac{1}{9} \ln|e^x - 3| + k = \\
&= -\frac{1}{9} x + \frac{1}{3e^x} + \frac{1}{9} \ln|e^x - 3| + k
\end{aligned}$$

$$c) \int \frac{e^{3x} - e^x}{e^{2x} + 1} dx$$

$$\text{Cambio: } e^x = t \rightarrow x = \ln t \rightarrow dx = \frac{1}{t} dt$$

$$\begin{aligned} \int \frac{e^{3x} - e^x}{e^{2x} + 1} dx &= \int \frac{t^3 - t}{t^2 + 1} \cdot \frac{1}{t} dt = \int \frac{t^2 - 1}{t^2 + 1} dt = \int \left(1 - \frac{2}{t^2 + 1} \right) dt = \\ &= t - 2 \operatorname{arc} \operatorname{tg} t + k = e^x - 2 \operatorname{arc} \operatorname{tg} (e^x) + k \end{aligned}$$

$$d) \int \frac{1}{1 + \sqrt{x}} dx$$

$$\text{Cambio: } x = t^2 \rightarrow dx = 2t dt$$

$$\begin{aligned} \int \frac{1}{1 + \sqrt{x}} dx &= \int \frac{2t dt}{1 + t} = \int \left(2 - \frac{2}{1 + t} \right) dt = 2t - 2 \ln |1 + t| + k = \\ &= 2\sqrt{x} - 2 \ln (1 + \sqrt{x}) + k \end{aligned}$$

Ejercicio 30.-

Encuentra la primitiva de $f(x) = \frac{1}{1 + 3x}$ que se anula para $x = 0$.

$$F(x) = \int \frac{1}{1 + 3x} dx = \frac{1}{3} \int \frac{3}{1 + 3x} dx = \frac{1}{3} \ln |1 + 3x| + k$$

$$F(0) = k = 0$$

$$\text{Por tanto: } F(x) = \frac{1}{3} \ln |1 + 3x|$$

Ejercicio 31.-

Halla la función F para la que $F'(x) = \frac{1}{x^2}$ y $F(1) = 2$.

$$F(x) = \int \frac{1}{x^2} dx = \frac{-1}{x} + k$$

$$F(1) = -1 + k = 2 \Rightarrow k = 3$$

$$\text{Por tanto: } F(x) = \frac{-1}{x} + 3$$

Ejercicio 32.-

De todas las primitivas de la función $y = 4x - 6$, ¿cuál de ellas toma el valor 4 para $x = 1$?

$$F(x) = \int (4x - 6) dx = 2x^2 - 6x + k$$

$$F(1) = 2 - 6 + k = 4 \Rightarrow k = 8$$

$$\text{Por tanto: } F(x) = 2x^2 - 6x + 8$$

Ejercicio 33.-

Halla $f(x)$ sabiendo que $f''(x) = 6x$, $f'(0) = 1$ y $f(2) = 5$.

$$\left. \begin{array}{l} f'(x) = \int 6x \, dx = 3x^2 + c \\ f'(0) = c = 1 \end{array} \right\} f'(x) = 3x^2 + 1$$

$$\left. \begin{array}{l} f(x) = \int (3x^2 + 1) \, dx = x^3 + x + k \\ f(2) = 10 + k = 5 \Rightarrow k = -5 \end{array} \right\}$$

Por tanto: $f(x) = x^3 + x - 5$

Ejercicio 34.-

Resuelve las siguientes integrales por sustitución:

a) $\int \frac{e^x}{1 - \sqrt{e^x}} \, dx$

b) $\int \sqrt{e^x - 1} \, dx$

• a) Haz $\sqrt{e^x} = t$. b) Haz $\sqrt{e^x - 1} = t$.

a) $\int \frac{e^x}{1 - \sqrt{e^x}} \, dx$

Cambio: $\sqrt{e^x} = t \rightarrow e^{x/2} = t \rightarrow \frac{x}{2} = \ln t \rightarrow dx = \frac{2}{t} \, dt$

$$\begin{aligned} \int \frac{e^x}{1 - \sqrt{e^x}} \, dx &= \int \frac{t^2 \cdot (2/t) \, dt}{1 - t} = \int \frac{2t \, dt}{1 - t} = \int \left(-2 + \frac{2}{1 - t} \right) dt = \\ &= -2t - 2 \ln |1 - t| + k = -2\sqrt{e^x} - 2 \ln |1 - \sqrt{e^x}| + k \end{aligned}$$

b) $\int \sqrt{e^x - 1} \, dx$

Cambio: $\sqrt{e^x - 1} = t \rightarrow e^x = t^2 + 1 \rightarrow x = \ln(t^2 + 1) \rightarrow dx = \frac{2t}{t^2 + 1} \, dt$

$$\begin{aligned} \int \sqrt{e^x - 1} \, dx &= \int t \cdot \frac{2t}{t^2 + 1} \, dt = \int \frac{2t^2}{t^2 + 1} \, dt = \int \left(2 - \frac{2}{t^2 + 1} \right) dt = \\ &= 2t - 2 \operatorname{arc} \operatorname{tg} t + k = 2\sqrt{e^x - 1} - 2 \operatorname{arc} \operatorname{tg} \sqrt{e^x - 1} + k \end{aligned}$$

Ejercicio 35.-

Determina la función $f(x)$ sabiendo que:

$$f''(x) = x \ln x, \quad f'(1) = 0 \quad \text{y} \quad f(e) = \frac{e}{4}$$

$$f'(x) = \int x \ln x \, dx$$

Integramos por partes:

$$\begin{cases} u = \ln x \rightarrow du = \frac{1}{x} dx \\ dv = x dx \rightarrow v = \frac{x^2}{2} \end{cases}$$

$$\left. \begin{aligned} f'(x) &= \frac{x^2}{2} \ln x - \int \frac{x}{2} dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + k = \frac{x^2}{2} \left(\ln x - \frac{1}{2} \right) + k \\ f'(1) &= \frac{1}{2} \left(-\frac{1}{2} \right) + k = -\frac{1}{4} + k = 0 \Rightarrow k = \frac{1}{4} \end{aligned} \right\}$$

$$f'(x) = \frac{x^2}{2} \left(\ln x - \frac{1}{2} \right) + \frac{1}{4}$$

$$f(x) = \int \left[\frac{x^2}{2} \left(\ln x - \frac{1}{2} \right) + \frac{1}{4} \right] dx = \underbrace{\int \frac{x^2}{2} \left(\ln x - \frac{1}{2} \right) dx}_I + \frac{1}{4} x$$

$$\begin{cases} u = \left(\ln x - \frac{1}{2} \right) \rightarrow du = \frac{1}{x} dx \\ dv = \frac{x^2}{2} dx \rightarrow v = \frac{x^3}{6} \end{cases}$$

$$I = \frac{x^3}{6} \left(\ln x - \frac{1}{2} \right) - \int \frac{x^2}{6} dx = \frac{x^3}{6} \left(\ln x - \frac{1}{2} \right) - \frac{x^3}{18} + k$$

Por tanto:

$$\left. \begin{aligned} f(x) &= \frac{x^3}{6} \left(\ln x - \frac{1}{2} \right) - \frac{x^3}{18} + \frac{1}{4} x + k \\ f(e) &= \frac{e^3}{12} - \frac{e^3}{18} + \frac{e}{4} + k = \frac{e^3}{36} + \frac{e}{4} + k = \frac{e}{4} \Rightarrow k = -\frac{e^3}{36} \end{aligned} \right\}$$

$$f(x) = \frac{x^3}{6} \left(\ln x - \frac{1}{2} \right) - \frac{x^3}{18} + \frac{1}{4} x - \frac{e^3}{36}$$

Ejercicio 36.-

Calcula la expresión de una función $f(x)$ tal que $f'(x) = x e^{-x^2}$ y que $f(0) = \frac{1}{2}$.

$$f(x) = \int x e^{-x^2} dx = -\frac{1}{2} \int -2x e^{-x^2} dx = -\frac{1}{2} e^{-x^2} + k$$

$$f(0) = -\frac{1}{2} + k = \frac{1}{2} \Rightarrow k = 1$$

Por tanto: $f(x) = -\frac{1}{2} e^{-x^2} + 1$

Ejercicio 37.-

Encuentra la función derivable $f: [-1, 1] \rightarrow \mathbb{R}$ que cumple $f(1) = -1$ y

$$f'(x) = \begin{cases} x^2 - 2x & \text{si } -1 \leq x < 0 \\ e^x - 1 & \text{si } 0 \leq x \leq 1 \end{cases}$$

- Si $x \neq 0$:

$$f(x) = \begin{cases} \frac{x^3}{3} - x^2 + k & \text{si } -1 \leq x < 0 \\ e^x - x + c & \text{si } 0 < x \leq 1 \end{cases}$$

- Hallamos k y c teniendo en cuenta que $f(1) = -1$ y que $f(x)$ ha de ser continua en $x = 0$.

$$f(1) = -1 \Rightarrow e - 1 + c = -1 \Rightarrow c = -e$$

$$\left. \begin{array}{l} \lim_{x \rightarrow 0^-} f(x) = k \\ \lim_{x \rightarrow 0^+} f(x) = 1 - e \end{array} \right\} k = 1 - e$$

$$\text{Por tanto: } f(x) = \begin{cases} \frac{x^3}{3} - x^2 + 1 - e & \text{si } -1 \leq x < 0 \\ e^x - x - e & \text{si } 0 \leq x \leq 1 \end{cases}$$

Ejercicio 38.-

De una función derivable se sabe que pasa por el punto $A(-1, -4)$ y que su derivada es:

$$f'(x) = \begin{cases} 2 - x & \text{si } x \leq 1 \\ 1/x & \text{si } x > 1 \end{cases}$$

- a) Halla la expresión de $f(x)$.
b) Obtén la ecuación de la recta tangente a $f(x)$ en $x = 2$.

- a) Si $x \neq 1$:

$$f(x) = \begin{cases} 2x - \frac{x^2}{2} + k & \text{si } x < 1 \\ \ln x + c & \text{si } x > 1 \end{cases}$$

Hallamos k y c teniendo en cuenta que $f(-1) = -4$ y que $f(x)$ ha de ser continua en $x = 1$.

$$f(-1) = -\frac{5}{2} + k = -4 \Rightarrow k = -\frac{3}{2}$$

$$\left. \begin{array}{l} \lim_{x \rightarrow 1^-} f(x) = \frac{3}{2} - \frac{3}{2} = 0 \\ \lim_{x \rightarrow 1^+} f(x) = c \end{array} \right\} c = 0$$

$$\text{Por tanto: } f(x) = \begin{cases} 2x - \frac{x^2}{2} - \frac{3}{2} & \text{si } x < 1 \\ \ln x & \text{si } x \geq 1 \end{cases}$$

b) $f(2) = \ln 2$; $f'(2) = \frac{1}{2}$

La ecuación de la recta tangente será: $y = \ln 2 + \frac{1}{2}(x - 2)$