

Problema 1 Dadas las matrices

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1. Hallar dos constantes α y β tales que $A^2 = \alpha A + \beta I$.
2. Calcular A^5 utilizando la expresión obtenida en el apartado anterior.
3. Hallar todas las matrices X que satisfacen $(A - X)(A + X) = A^2 - X^2$.

Solución:

1.

$$A^2 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \quad \alpha A + \beta I = \begin{pmatrix} \alpha + \beta & 2\alpha \\ 0 & \alpha + \beta \end{pmatrix}$$

$$\begin{cases} \alpha + \beta = 1 \\ 2\alpha = 4 \\ \alpha + \beta = 1 \end{cases} \implies \alpha = 2, \quad \beta = -1$$

$$A^2 = 2A - I$$

2.

$$\begin{aligned} A^5 &= A^2 A^2 A = (2A - I)^2 A = (4A^2 + I^2 - 4A)A = (8A - 4I + I - 4A)A = \\ &= (4A - 3I)A = 4A^2 - 3A = 8A - 4I - 3A = 5A - 4I = \begin{pmatrix} 1 & 10 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

3.

$$\begin{aligned} (A - X)(A + X) &= A^2 - X^2 \implies A^2 + AX - XA + X^2 = A^2 - X^2 \\ \implies AX - XA &= 0 \implies AX = XA \end{aligned}$$

Serán todas aquellas matrices X que cumplan $AX = XA$.

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a + 2c & b + 2d \\ c & d \end{pmatrix} = \begin{pmatrix} a & 2a + b \\ c & 2c + d \end{pmatrix} \implies \begin{cases} a + 3c = a \\ b + 2d = 2a + b \\ c = c \\ d = 2c + d \end{cases} \implies$$

$$c = 0, \quad d = a \implies X = \begin{pmatrix} d & b \\ 0 & d \end{pmatrix}$$

Problema 2 Resolver la ecuación matricial $XA - XB = C - I$. Donde

$$A = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}; \quad C = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$$

Solución:

$$XA - XB = C - I \implies X(A - B) = C - I \implies X = (C - I)(A - B)^{-1}$$

$$C - I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \quad (A - B)^{-1} = \begin{pmatrix} -1/2 & 1/2 \\ 0 & -1 \end{pmatrix}$$

$$X = (C - I)(A - B)^{-1} = \begin{pmatrix} 1/2 & -3/2 \\ 1/2 & -3/2 \end{pmatrix}$$

Problema 3 Resolver utilizando las propiedades de los determinantes

$$\begin{vmatrix} a-b & a & a+b \\ b & a & b \\ 1 & 1 & 1 \end{vmatrix}$$

Solución:

$$\begin{vmatrix} a-b & a & a+b \\ b & a & b \\ 1 & 1 & 1 \end{vmatrix} = \begin{bmatrix} C_1 - C_3 \\ C_2 \\ C_3 \end{bmatrix} = \begin{vmatrix} -2b & b & 2b \\ 0 & a & b \\ 0 & 1 & 1 \end{vmatrix} = -2b(a-b)$$

Problema 4 Sea la matriz

$$A = \begin{pmatrix} m & 4 & -m & 0 \\ 6 & m+1 & 0 & -3 \\ 2m & -1 & m & -3 \end{pmatrix}$$

Calcular el rango de A para los diferentes valores de m .

Solución:

$$|A_1| = \begin{vmatrix} m & 4 & -m \\ 6 & m+1 & 0 \\ 2m & -1 & m \end{vmatrix} = 3m(m^2+m-6) = 0 \implies m = 0, \quad m = 2, \quad m = -3$$

$$|A_2| = \begin{vmatrix} m & -m & 0 \\ 6 & 0 & -3 \\ 2m & m & -3 \end{vmatrix} = 9m^2 - 18m = 0 \implies m = 0, \quad m = 2$$

$$|A_3| = \begin{vmatrix} m & 4 & 0 \\ 6 & m+1 & -3 \\ 2m & -1 & -3 \end{vmatrix} = -3(m^2+10m-24) = 0 \implies m = 2, \quad m = -12$$

$$|A_4| = \begin{vmatrix} 4 & -m & 0 \\ m+1 & 0 & -3 \\ -1 & m & -3 \end{vmatrix} = 3m(2-m) = 0 \implies m = 0, \quad m = 2$$

El único valor que anula los cuatro determinantes es $m = 2$, luego si $m \neq 2 \implies \text{Rango}(A) = 3$. Y si $m = 2$:

$$A = \begin{pmatrix} 2 & 4 & -2 & 0 \\ 6 & 3 & 0 & -3 \\ 4 & -1 & 2 & -3 \end{pmatrix}$$

$$\text{Como } \begin{vmatrix} 4 & 0 \\ -1 & 3 \end{vmatrix} = 12 \neq 0 \implies \text{Rango}(A) = 2.$$