

Calcular $\int \frac{x+1}{x^3+x^2-6x} dx$

Factorizamos el denominador

$$x^3+x^2-6x=0 \Rightarrow x(x^2+x-6)=0 \Rightarrow \begin{cases} x=0 \\ x^2+x-6=0 \Rightarrow x=\begin{Bmatrix} 2 \\ -3 \end{Bmatrix} \end{cases}$$

$$x^3+x^2-6x = x(x+3)(x-2)$$

Descomposición en fracciones simples

$$\begin{aligned} \frac{x+1}{x^3+x^2-6x} &= \frac{x+1}{x(x+3)(x-2)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-2} = \\ &= \frac{A(x+3)(x-2) + Bx(x-2) + Cx(x+3)}{x(x+3)(x-2)} \end{aligned}$$

Para determinar A, B y C le damos a x los valores 0, 2 y -3:

$$\begin{aligned} x=0 &\Rightarrow (\text{igualando los numeradores}) 0+1 = A(0+3)(0-2) + B \cdot 0(0-2) + C \cdot 0(0+3) \Rightarrow \\ &\Rightarrow A = -\frac{1}{6} \end{aligned}$$

$$x=-3 \Rightarrow B = -\frac{2}{15}$$

$$x=2 \Rightarrow C = \frac{3}{10}$$

$$\text{Por tanto: } \frac{x+1}{x^3+x^2-6x} = \frac{-\frac{1}{6}}{x} + \frac{-\frac{2}{15}}{x+3} + \frac{\frac{3}{10}}{x-2}$$

Calculamos la integral:

$$\int \frac{x+1}{x^3+x^2-6x} dx = -\frac{1}{6} \int \frac{1}{x} dx - \frac{2}{15} \int \frac{1}{x+3} dx + \frac{3}{10} \int \frac{1}{x-2} dx =$$

$$\boxed{= -\frac{1}{6} \ln|x| - \frac{2}{15} \ln|x+3| + \frac{3}{10} \ln|x-2| + C}$$

Calcular $\int \frac{3x}{x^2+2x+3} dx$

Factorizamos el denominador:

$$x^2+2x+3=0 \Rightarrow x = \frac{-2 \pm \sqrt{4-4 \cdot 1 \cdot 3}}{2 \cdot 1} = \frac{-2 \pm \sqrt{-8}}{2} = \frac{-2 \pm 2i\sqrt{2}}{2} = -1 \pm i\sqrt{2}$$

La vamos a resolver de otra forma:

$$\int \frac{3x}{x^2+2x+3} dx = \int \frac{3x+6-6}{x^2+2x+3} dx = \frac{1}{2} \int \frac{6x+6-6}{x^2+2x+3} dx =$$

↑
Vamos buscando en el numerador
la derivada del denominador

$$= \frac{1}{2} \int \frac{6x+6}{x^2+2x+3} dx - \frac{6}{2} \int \frac{1}{x^2+2x+3} dx =$$

$$= \frac{3}{2} \underbrace{\int \frac{2x+2}{x^2+2x+3} dx}_{\text{inmediata}} - \frac{6}{2} \underbrace{\int \frac{1}{x^2+2x+3} dx}_{\text{cambio de variable}}$$

Calculamos $\int \frac{1}{x^2+2x+3} dx$

$$\int \frac{1}{x^2+2x+3} dx = \int \frac{1}{x^2+2x+1+2} dx = \int \frac{1}{(x+1)^2+2} dx =$$

$$= \left[\begin{matrix} t = x+1 \\ dt = dx \end{matrix} \right] = \int \frac{1}{t^2+2} dt = \int \frac{1}{2\left(\frac{t^2}{2}+1\right)} dt =$$

$$= \frac{1}{2} \int \frac{1}{\left(\frac{t}{\sqrt{2}}\right)^2+1} dt = \frac{1}{2} \sqrt{2} \int \frac{\frac{1}{\sqrt{2}}}{\left(\frac{t}{\sqrt{2}}\right)^2+1} dt = \frac{\sqrt{2}}{2} \operatorname{arctg} \left(\frac{t}{\sqrt{2}} \right) =$$

$$= \frac{\sqrt{2}}{2} \operatorname{arctg} \left(\frac{x+1}{\sqrt{2}} \right)$$

Por tanto:

$$\int \frac{3x}{x^2+2x+3} dx = \frac{3}{2} \int \frac{2x+2}{x^2+2x+3} dx - \frac{6}{2} \int \frac{1}{x^2+2x+3} dx =$$

$$= \frac{3}{2} \ln |x^2+2x+3| - \frac{3\sqrt{2}}{2} \operatorname{arctg} \left(\frac{x+1}{\sqrt{2}} \right) + C$$

Calcular $\int \frac{x^4 - 3x^3 - 3x - 2}{x^3 - x^2 - 2x} dx$

Efectuamos la división:

$$\begin{array}{r} x^4 - 3x^3 \quad -3x - 2 \\ -x^4 + x^3 + 2x^2 \\ \hline -2x^3 + 2x^2 - 3x - 2 \\ 2x^3 - 2x^2 - 4x \\ \hline -7x - 2 \end{array} \quad \begin{array}{r} x^3 - x^2 - 2x \\ x - 2 \end{array}$$

$$\frac{x^4 - 3x^3 - 3x - 2}{x^3 - x^2 - 2x} = x - 2 + \frac{-7x - 2}{x^3 - x^2 - 2x}$$

Factorizamos el denominador:

$$x^3 - x^2 - 2x = 0 \Rightarrow x^3 - x^2 - 2x = x(x^2 - x - 2) = 0 \Rightarrow \begin{cases} x = 0 \\ x^2 - x - 2 = 0 \end{cases}$$

$$\Rightarrow x = \begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$$

$$x^3 - x^2 - 2x = x(x-2)(x+1)$$

Expresamos en fracciones simples la fracción $\frac{-7x-2}{x^3-x^2-2x}$

$$\begin{aligned} \frac{-7x-2}{x^3-x^2-2x} &= \frac{-7x-2}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1} = \\ &= \frac{A(x-2)(x+1) + Bx(x+1) + Cx(x-2)}{x(x-2)(x+1)} \end{aligned}$$

$$x=0 \Rightarrow (\text{igualando numeradores}) \quad -2A = -2 \Rightarrow A = 1$$

$$x=2 \Rightarrow 6B = -16 \Rightarrow B = -\frac{8}{3}$$

$$x=-1 \Rightarrow 3C = 5 \Rightarrow C = \frac{5}{3}$$

$$\frac{-7x-2}{x^3-x^2-2x} = \frac{1}{x} + \frac{-\frac{8}{3}}{x-2} + \frac{\frac{5}{3}}{x+1}$$

Calculamos:

$$\begin{aligned} \int \frac{x^4 - 3x^3 - 3x - 2}{x^3 - x^2 - 2x} dx &= \int (x-2) dx + \int \frac{-7x-2}{x^3-x^2-2x} dx = \frac{x^2}{2} - 2x + \\ &+ \int \frac{1}{x} dx - \frac{8}{3} \int \frac{1}{x-2} dx + \frac{5}{3} \int \frac{1}{x+1} dx = \\ &= \boxed{\frac{x^2}{2} - 2x + \ln|x| - \frac{8}{3} \ln|x-2| + \frac{5}{3} \ln|x+1| + C} \end{aligned}$$

Calcular $\int \frac{6x+10}{-x^3+x^2+x-1} dx$

Factorizamos el denominador:

$$-x^3+x^2+x-1 = (x-1)(x+1)\underline{(-x+1)} = -(x+1)(x-1)(x-1) = -(x+1)(x-1)^2$$

	-1	1	1	-1
1		-1	0	1
	-1	0	1	0
-1		1	-1	
	-1	1	0	

Cuidado al factorizar que el coeficiente de x^3 es -1 .

Recuerda: $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

Raíces: $x_0, \dots, x_n$

Factorización: $P(x) = a_n (x-x_0) \dots (x-x_n)$

Descomposición en fracciones simples:

$$\begin{aligned} \frac{6x+10}{-x^3+x^2+x-1} &= \frac{6x+10}{-(x+1)(x-1)^2} = \frac{A}{-(x+1)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} = \\ &= \frac{A(x-1)^2 - B(x+1)(x-1) - C(x+1)}{-(x+1)(x-1)^2} \end{aligned}$$

$$\begin{cases} \text{Raíces: } x=1 \Rightarrow (\text{igualando numeradores}) & 16 = -2C \Rightarrow C = -8 \\ & x=-1 \Rightarrow 4 = 4A \Rightarrow A = 1 \\ \text{Valor: } x=0 \Rightarrow 10 = A+B-C \Rightarrow B = 1 \end{cases}$$

$$\frac{6x+10}{-x^3+x^2+x-1} = \frac{-6x+10}{-(x+1)(x-1)^2} = \frac{1}{-(x+1)} + \frac{1}{(x-1)} + \frac{-8}{(x-1)^2}$$

Calculamos:

$$\int \frac{6x+10}{-x^3+x^2+x-1} dx = - \int \frac{1}{x+1} dx + \int \frac{1}{x-1} dx - 8 \int \frac{1}{(x-1)^2} dx =$$

$$= -\ln|x+1| + \ln|x-1| - 8 \int (x-1)^{-2} dx =$$

$$= -\ln|x+1| + \ln|x-1| - 8 \frac{(x-1)^{-2+1}}{-2+1} + C$$

$$= -\ln|x+1| + \ln|x-1| + 8 \frac{1}{x-1} + C$$

Resuelve $\int \frac{x^2-1}{x(x^2+1)} dx$

Factorización del denominador:

$$x(x^2+1)=0 \Rightarrow \begin{cases} x=0 \\ x^2+1=0 \Rightarrow x=\pm\sqrt{-1}=\pm i \in \mathbb{C} \end{cases}$$

Descomposición en fracciones simples:

$$\frac{x^2-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1) + (Bx+C)x}{x(x^2+1)} = \frac{Ax^2 + A + Bx^2 + Cx}{x(x^2+1)}$$

$$\Rightarrow x^2-1 = (A+B)x^2 + Cx + A \Rightarrow$$

$$\Rightarrow (\text{igualando los términos en}) \begin{cases} (x^2) : A+B=1 \\ (x) : C=0 \\ (\text{indep.}) : A=-1 \end{cases} \Rightarrow \begin{cases} A=-1 \\ B=2 \\ C=0 \end{cases}$$

$$\frac{x^2-1}{x(x^2+1)} = \frac{-1}{x} + \frac{2x}{x^2+1}$$

Resolvemos:

$$\int \frac{x^2-1}{x(x^2+1)} dx = - \int \frac{1}{x} dx + \int \frac{2x}{x^2+1} dx =$$

$$= -\ln|x| + \ln|x^2+1| + C$$

Calcula $\int \frac{x+2}{x^3-4x^2+4x} dx$

Factorizamos el denominador:

$$x^3-4x^2+4x = x(x^2-4x+4) = 0 \Rightarrow \begin{cases} x=0 \\ x^2-4x+4=0 \Rightarrow x=2 \text{ (doble)} \end{cases}$$

$$x^3-4x^2+4x = x(x-2)^2$$

Descomponemos en fracciones simples:

$$\begin{aligned} \frac{x+2}{x^3-4x^2+4x} &= \frac{x+2}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2} = \\ &= \frac{A(x-2)^2 + Bx(x-2) + Cx}{x(x-2)^2} \end{aligned}$$

$$x+2 = A(x-2)^2 + Bx(x-2) + Cx \Rightarrow \left\{ \begin{array}{l} x=2 \Rightarrow 2C=4 \Rightarrow C=2 \\ x=0 \Rightarrow 4A=2 \Rightarrow A=\frac{1}{2} \\ x=1 \Rightarrow 3=A-B+C \end{array} \right\} \Rightarrow$$

$$\Rightarrow \left\{ \begin{array}{l} A=\frac{1}{2} \\ B=-\frac{1}{2} \\ C=2 \end{array} \right. \Rightarrow \frac{x+2}{x^3-4x^2+4x} = \frac{\frac{1}{2}}{x} + \frac{-\frac{1}{2}}{x-2} + \frac{2}{(x-2)^2}$$

Resolvemos:

$$\int \frac{x+2}{x^3-4x^2+4x} dx = \frac{1}{2} \int \frac{1}{x} dx - \frac{1}{2} \int \frac{1}{x-2} dx + 2 \int \frac{1}{(x-2)^2} dx =$$

$$= \frac{1}{2} \ln|x| - \frac{1}{2} \ln|x-2| + 2 \int (x-2)^{-2} dx =$$

$$= \frac{1}{2} \ln|x| - \frac{1}{2} \ln|x-2| + 2 \frac{(x-2)^{-2+1}}{-2+1} =$$

$$= \frac{1}{2} \ln|x| - \frac{1}{2} \ln|x-2| - \frac{2}{(x-2)} + C$$

Resuelve $\int \frac{x^2 - x + 1}{x^3 + x} dx$

Factorizamos el denominador:

$$x^3 + x = x(x^2 + 1) = 0 \Rightarrow \begin{cases} x = 0 \\ x^2 + 1 = 0 \Rightarrow x = \pm\sqrt{-1} = \pm i \in \mathbb{C} \end{cases}$$

Descomponemos en fracciones simples:

$$\begin{aligned} \frac{x^2 - x + 1}{x^3 + x} &= \frac{x^2 - x + 1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} = \frac{A(x^2 + 1) + x(Bx + C)}{x(x^2 + 1)} = \\ &= \frac{Ax^2 + A + Bx^2 + Cx}{x(x^2 + 1)} \end{aligned}$$

$$x^2 - x + 1 = (A + B)x^2 + Cx + A \Rightarrow$$

$$\Rightarrow (\text{igualando términos en}) \begin{cases} (x^2): A + B = 1 \\ (x): C = -1 \\ (\text{indep.}): A = 1 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} A = 1 \\ B = 0 \\ C = -1 \end{cases} \Rightarrow \frac{x^2 - x + 1}{x^3 + x} = \frac{1}{x} + \frac{-1}{x^2 + 1}$$

Resolvemos:

$$\int \frac{x^2 - x + 1}{x^3 + x} dx = \int \frac{1}{x} dx - \int \frac{1}{x^2 + 1} dx =$$

$$= \ln|x| - \arctg x + C$$

Calcula $\int \frac{x^2-2}{x^3-3x+2} dx$

Factorizamos el denominador:

$$x^3-3x+2 = (x-1)^2(x+2)$$

$$\begin{array}{r|rrrr} & 1 & 0 & -3 & 2 \\ 1 & & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & 0 \\ 1 & & 1 & 2 & \\ \hline & 1 & 2 & 0 & \end{array}$$

Descomponemos en fracciones simples:

$$\begin{aligned} \frac{x^2-2}{x^3-3x+2} &= \frac{x^2-2}{(x+2)(x-1)^2} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2} = \\ &= \frac{A(x-1)^2 + B(x+2)(x-1) + C(x+2)}{(x+2)(x-1)^2} \end{aligned}$$

Igualemos los numeradores y le damos valores a x:

$$\left. \begin{aligned} x=1 &\Rightarrow -2 = 3C \Rightarrow C = -\frac{1}{3} \\ x=0 &\Rightarrow -2 = A - 2B + 2C \\ x=-1 &\Rightarrow -1 = 4A - 2B + C \end{aligned} \right\} \begin{aligned} -2 &= A - 2B - \frac{2}{3} \\ -1 &= 4A - 2B - \frac{1}{3} \end{aligned} \right\} \begin{aligned} A &= \frac{2}{9} \\ B &= \frac{7}{9} \end{aligned}$$

$$\frac{x^2-2}{x^3-3x+2} = \frac{\frac{2}{9}}{x+2} + \frac{\frac{7}{9}}{x-1} + \frac{-\frac{1}{3}}{(x-1)^2}$$

Resolvemos:

$$\begin{aligned} \int \frac{x^2-2}{x^3-3x+2} dx &= \frac{2}{9} \int \frac{1}{x+2} dx + \frac{7}{9} \int \frac{1}{x-1} dx - \frac{1}{3} \int \frac{1}{(x-1)^2} dx = \\ &= \frac{2}{9} \ln|x+2| + \frac{7}{9} \ln|x-1| - \frac{1}{3} \int (x-1)^{-2} dx = \\ &= \frac{2}{9} \ln|x+2| + \frac{7}{9} \ln|x-1| - \frac{1}{3} \frac{(x-1)^{-2+1}}{-2+1} = \\ &= \frac{2}{9} \ln|x+2| + \frac{7}{9} \ln|x-1| + \frac{1}{3(x-1)} + C \end{aligned}$$

Calcula $\int (x^2 + 2x + 1) \ln(x) dx$

Se resuelve integrando por partes:

$$\int (x^2 + 2x + 1) \ln(x) dx = \left[u = \ln x \rightarrow du = \frac{1}{x} dx \right. \\ \left. dv = (x^2 + 2x + 1) dx \rightarrow v = \frac{x^3}{3} + x^2 + x \right] =$$

$$= \left(\frac{x^3}{3} + x^2 + x \right) \ln x - \int \left(\frac{x^3}{3} + x^2 + x \right) \frac{1}{x} dx =$$

$$= \left(\frac{x^3}{3} + x^2 + x \right) \ln x - \int \left(\frac{x^2}{3} + x + 1 \right) dx =$$

$$= \left(\frac{x^3}{3} + x^2 + x \right) \ln x - \frac{x^3}{9} - \frac{x^2}{2} - x + C$$

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Calcula $\int_e^{e^3} \frac{\log x}{x} dx$

Igual que en el ejercicio anterior
 $\log x$ denota el logaritmo neperiano
 (Ante la duda, se pregunta)

En primer lugar calculamos $\int \frac{\log x}{x} dx$:

$$\int \frac{\ln x}{x} dx = \left[t = \ln x \rightarrow dt = \frac{1}{x} dx \right] =$$

$$= \int t dt = \frac{t^2}{2} = \frac{\ln^2 x}{2} + C$$

Aplicamos la regla de Barrow:

$$\int_e^{e^3} \frac{\ln x}{x} dx = \frac{\ln^2 x}{2} \Big|_e^{e^3} = \frac{\ln^2 e^3}{2} - \frac{\ln^2 e}{2} = \frac{1}{2} (3^2 - 1^2) = \frac{8}{2} = 4$$

Resuelve la integral $\int \frac{x^2+1}{x^3-3x+2} dx$

Factorizamos el denominador:

$$x^3-3x+2 = (x-1)^2(x+2)$$

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -3 & 2 \\ 1 & & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & 0 \\ 1 & & 1 & 2 & \\ \hline & 1 & 2 & 0 & \end{array}$$

Descomponemos en fracciones simples:

$$\begin{aligned} \frac{x^2+1}{x^3-3x+2} &= \frac{x^2+1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2} = \\ &= \frac{A(x-1)(x+2) + B(x+2) + C(x-1)^2}{(x-1)^2(x+2)} \end{aligned}$$

Igualemos los numeradores

$$x^2+1 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

y le damos valores a x:

$$\begin{cases} x=1 : 3B = 2 \Rightarrow B = \frac{2}{3} \\ x=-2 : 9C = 5 \Rightarrow C = \frac{5}{9} \\ x=0 : -2A+2B+C = 1 \Rightarrow A = \frac{4}{9} \end{cases}$$

$$\frac{x^2+1}{x^3-3x+2} = \frac{\frac{4}{9}}{x-1} + \frac{\frac{2}{3}}{(x-1)^2} + \frac{\frac{5}{9}}{x+2}$$

Resolvemos:

$$\int \frac{x^2+1}{x^3-3x+2} dx = \frac{4}{9} \int \frac{1}{x-1} dx + \frac{2}{3} \int \frac{1}{(x-1)^2} dx + \frac{5}{9} \int \frac{1}{x+2} dx =$$

$$= \frac{4}{9} \ln|x-1| + \frac{2}{3} \frac{(x-1)^{-2+1}}{-2+1} + \frac{5}{9} \ln|x+2| =$$

$$= \frac{4}{9} \ln|x-1| - \frac{2}{3(x-1)} + \frac{5}{9} \ln|x+2| + C$$