

Cambio de variable

$$\textcircled{1} \int \frac{2^{\ln x}}{x} dx = \left[\begin{array}{l} \ln x = t \\ \frac{1}{x} dx = dt \end{array} \right] = \int 2^t dt = \frac{2^t}{\ln 2} + C = \frac{2^{\ln x}}{\ln 2} + C$$

$$\textcircled{2} \int 3x \cos x^2 dx = \left[\begin{array}{l} x^2 = t \\ 2x dx = dt \\ dx = \frac{dt}{2x} \end{array} \right] = \int 3x \cos t \frac{dt}{2x} = \frac{3}{2} \int \cos t dt = \frac{3}{2} \operatorname{sen} t = \\ = \frac{3}{2} \operatorname{sen} x^2 + C$$

$$\textcircled{3} \int \frac{\operatorname{tg} x}{\cos^2 x} dx = \left[\begin{array}{l} \operatorname{tg} x = t \\ \frac{1}{\cos^2 x} dx = dt \end{array} \right] = \int t dt = \frac{1}{2} t^2 = \frac{\operatorname{tg}^2 x}{2} + C$$

$$\textcircled{4} \int \frac{\operatorname{sen} x \cos x}{1 + \operatorname{sen}^4 x} dx = \left[\begin{array}{l} \operatorname{sen}^2 x = t \\ 2 \operatorname{sen} x \cos x dx = dt \end{array} \right] = \frac{1}{2} \int \frac{dt}{1+t^2} = \frac{1}{2} \operatorname{arctg} t = \\ = \frac{1}{2} \operatorname{arctg}(\operatorname{sen}^2 x) + C$$

$$\textcircled{5} \int \frac{\cos \sqrt{2x}}{\sqrt{2x}} dx = \left[\begin{array}{l} \sqrt{2x} = t \\ \frac{1}{\sqrt{2x}} dx = dt \end{array} \right] = -\frac{1}{2} \int \cos t dt = -\operatorname{sen} t = -\operatorname{sen} \sqrt{2x} + C$$

Por partes

$$\textcircled{6} \int x \cos x dx = \left[\begin{array}{l} u = x \rightarrow du = dx \\ dv = \cos x dx \rightarrow v = \operatorname{sen} x \end{array} \right] = x \operatorname{sen} x - \int \operatorname{sen} x dx = x \operatorname{sen} x + \cos x + C$$

$$\textcircled{7} \int x^2 e^x dx = \left[\begin{array}{l} u = x^2 \rightarrow du = 2x dx \\ dv = e^x dx \rightarrow v = e^x \end{array} \right] = x^2 e^x - \int 2x e^x dx \quad \textcircled{1}$$

Calculamos $\int x e^x dx$ por partes:

$$\int x e^x dx = \left[\begin{array}{l} u = x \rightarrow du = dx \\ dv = e^x dx \rightarrow v = e^x \end{array} \right] = x e^x - \int e^x dx = x e^x - e^x$$

$$\textcircled{1} x^2 e^x - 2(x e^x - e^x) = e^x(x^2 - 2x + 2) + C$$

$$\textcircled{8} \int \cos^2 x dx = \int \cos x \cos x dx = \left[\begin{array}{l} u = \cos x \rightarrow du = -\operatorname{sen} x dx \\ dv = \cos x dx \rightarrow v = \operatorname{sen} x \end{array} \right] =$$

$$= \cos x \operatorname{sen} x - \int -\operatorname{sen}^2 dx = \cos x \operatorname{sen} x + \int (1 - \cos^2 x) dx = \operatorname{sen} x \cos x + x - \int \cos^2 x dx \\ \operatorname{sen}^2 x + \cos^2 x = 1$$

$$\Rightarrow 2 \int \cos^2 x dx = \operatorname{sen} x \cos x + x \Rightarrow \int \cos^2 x dx = \frac{1}{2} (\operatorname{sen} x \cos x + x) + C$$

$$\textcircled{9} \int x^2 \cos x \, dx = \left[\begin{array}{l} u = x^2 \rightarrow du = 2x \, dx \\ dv = \cos x \, dx \rightarrow v = \text{sen } x \end{array} \right] = x^2 \text{sen } x - \int 2x \text{sen } x \, dx \quad \textcircled{1}$$

Calculamos $\int x \text{sen } x \, dx$ por partes

$$\int x \text{sen } x \, dx = \left[\begin{array}{l} u = x \rightarrow du = dx \\ dv = \text{sen } x \, dx \rightarrow v = -\cos x \end{array} \right] = -x \cos x + \int \cos x \, dx =$$

$$= -x^2 \cos x + \text{sen } x$$

$$\textcircled{1} \quad x^2 \text{sen } x - 2(-x^2 \cos x + \text{sen } x) = x^2 \text{sen } x + 2x^2 \cos x - 2 \text{sen } x + C$$

$$\textcircled{10} \int \frac{x}{2} \text{sen } 3x \, dx = \left[\begin{array}{l} \frac{x}{2} = u \rightarrow du = \frac{1}{2} dx \\ dv = \text{sen } 3x \, dx \rightarrow -\frac{1}{3} \cos 3x = v \end{array} \right] = -\frac{1}{6} x \cos 3x + \frac{1}{6} \int \cos 3x \, dx$$

$$= -\frac{1}{6} x \cos 3x + \frac{1}{18} \text{sen } 3x + C$$

$$\textcircled{11} \int x 3^x \, dx = \left[\begin{array}{l} u = x \rightarrow du = dx \\ dv = 3^x \, dx \rightarrow v = \frac{3^x}{\ln 3} \end{array} \right] = \frac{x 3^x}{\ln 3} - \int \frac{3^x}{\ln 3} \, dx =$$

$$= \frac{1}{\ln 3} (x 3^x) - \frac{1}{\ln 3} \int 3^x \, dx = \frac{x 3^x}{\ln 3} - \frac{1}{\ln 3} \frac{3^x}{\ln 3} = \frac{3^x}{\ln 3} \left(x - \frac{1}{\ln 3} \right) + C$$

Integración de funciones racionales

I Con solo raíces simples en el denominador

$$\textcircled{12} \int \frac{4x^2 + 8x + 6}{x^3 + 2x^2 - x - 2} \, dx$$

$$x^3 + 2x^2 - x - 2 = (x-1)(x+1)(x+2)$$

	1	2	-1	-2
1		1	3	2
	1	3	2	0
-1		-1	-2	
	1	2	0	
-2		-2		
	1	0		

$$\frac{4x^2 + 8x + 6}{x^3 + 2x^2 - x - 2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2} =$$

$$= \frac{A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1)}{(x-1)(x+1)(x+2)}$$

$$4x^2 + 8x + 6 = A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1) =$$

$$= x^2(A+B+C) + x(3A+B) + (2A-2B-C) \Rightarrow$$

$$\Rightarrow \begin{cases} 4 = A+B+C \\ 8 = 3A+B \\ 6 = 2A-2B-C \end{cases}$$

$$\rightarrow (A, B, C) = (3, -1, 2)$$

También se pueden calcular igualando los numeradores y sustituyendo el valor de las raíces del denominador.

$$\int \frac{4x^2 + 8x + 6}{x^3 + 2x^2 - x - 2} dx = \int \frac{3}{x-1} dx + \int \frac{-1}{x+1} dx + \int \frac{2}{x+2} dx =$$

$$= 3 \ln|x-1| - \ln|x+1| + 2 \ln|x+2| + C$$

$$(13) \int \frac{x+3}{x^2+3x+2} dx$$

$$x^2+3x+2 = (x+1)(x+2)$$

$$\begin{array}{r|rrrr} & 1 & 3 & 2 & \\ -1 & & -1 & -2 & \\ \hline & 1 & 2 & 0 & \end{array}$$

$$\frac{x+3}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{A(x+2) + B(x+1)}{(x+1)(x+2)}$$

$$\left. \begin{array}{l} x = -1 \rightarrow 2 = A \\ x = -2 \rightarrow 1 = -B \end{array} \right\} \frac{x+3}{x^2+3x+2} = \frac{2}{x+1} - \frac{1}{x+2}$$

$$\int \frac{x+3}{x^2+3x+2} dx = \int \frac{2}{x+1} dx - \int \frac{1}{x+2} dx = 2 \ln|x+1| - \ln|x+2| + C$$

II Con raíces múltiples en el denominador

$$(14) \int \frac{2x-1}{x^3-3x^2+3x-1} dx$$

$$x^3-3x^2+3x-1 = (x-1)^3$$

$$\begin{array}{r|rrrr} & 1 & -3 & 3 & -1 \\ 1 & & 1 & -2 & 1 \\ \hline & 1 & -2 & 1 & 0 \\ 1 & & 1 & -1 & \\ \hline & 1 & -1 & 0 & \end{array}$$

$$\begin{aligned} \frac{2x-1}{x^3-3x^2+3x-1} &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} = \\ &= \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^3} \end{aligned}$$

$$\text{Raíz: } x=1 \Rightarrow C=1$$

$$\text{Valores: } \left\{ \begin{array}{l} x=0 \rightarrow -1 = A-B \\ x=-1 \rightarrow 3 = A+B \end{array} \right\} \rightarrow (A, B) = (2, 2)$$

$$\frac{2x-1}{x^3-3x^2+3x-1} = \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{1}{(x-1)^3}$$

$$\int \frac{2x-1}{x^3-3x^2+3x-1} dx = \int \frac{1}{x-1} dx + 2 \int \frac{1}{(x-1)^2} dx + \int \frac{1}{(x-1)^3} dx =$$

$$= \ln|x-1| - \frac{2}{x-1} - \frac{1}{2(x-1)^2} + C$$

III) Con raíces simples y múltiples en el denominador

15) $\int \frac{3x+7}{x^3-x^2-x+1} dx$

$$x^3-x^2-x+1 = (x+1)(x-1)^2$$

	1	-1	-1	1
1		1	0	-1
	1	0	-1	0
1		1	1	
	1	1	0	

$$\frac{3x+7}{x^3-x^2-x+1} = \frac{A}{x+1} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} =$$

$$= \frac{A(x-1)^2 + B(x+1)(x-1) + C(x+1)}{(x+1)(x-1)^2}$$

Raíces: $x=1 \rightarrow C=5$

$x=-1 \rightarrow A=1$

Valor: $x=0 \rightarrow B=-1$

$$\frac{3x+7}{x^3-x^2-x+1} = \frac{1}{x+1} - \frac{1}{x-1} + \frac{5}{(x-1)^2}$$

$$\int \frac{3x+7}{x^3-x^2-x+1} dx = \int \frac{1}{x+1} dx - \int \frac{1}{x-1} dx + 5 \int \frac{1}{(x-1)^2} dx =$$

$$= \ln|x+1| - \ln|x-1| - \frac{5}{x-1} + C$$