

Calcular las siguientes integrales:

1. $\int_1^e \frac{1}{x(1+\ln x)} dx$ puedes hacer $t = \ln x$ (Extremadura Junio 2008)
2. $\int \frac{x+5}{x^2+4x+3} dx$ (Galicia Junio 2008)
3. $\int_0^{\sqrt{3}} \frac{2x^3}{\sqrt{x^2+1}} dx$ puedes hacer $t = \sqrt{x^2+1}$ (La Rioja Junio 2008)
4. $\int \frac{x^3-2x^2}{x^2-2x+1} dx$ (Murcia Junio 2008)
5. $\int \frac{1}{x^2-(a+1)x+a} dx$ donde se supone que a no es cero. (País Vasco Junio 2008)

Soluciones:

$$1. \quad \int \frac{1}{x(1+\ln x)} dx = \int \frac{1}{t+1} dt = \ln(t+1) = \ln(\ln x + 1) + C$$

$$\int_1^e \frac{1}{x(1+\ln x)} dx = \ln(\ln x + 1) \Big|_1^e = \ln 2$$

El cambio que hemos hecho es el siguiente:

$$t = \ln x \implies dt = \frac{1}{x} dx$$

$$2. \quad \int \frac{x+5}{x^2+4x+3} dx = \int \frac{x+5}{(x+1)(x+3)} dx$$

Hacemos la descomposición en fracciones simples:

$$\frac{x+5}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3} = \frac{A(x+3) + B(x+1)}{(x+1)(x+3)}$$

$$x+5 = A(x+3) + B(x+1)$$

$$\text{Si } x = -3 \implies B = -1$$

$$\text{Si } x = -1 \implies A = 2$$

$$\int \frac{x+5}{x^2+4x+3} dx = 2 \int \frac{1}{x+1} dx - \int \frac{1}{x+3} dx = 2 \ln|x+1| - \ln|x+3| + C$$

3.

$$\int \frac{2x^3}{\sqrt{x^2+1}} dx = 2 \int (t^2 - 1) = \frac{2t^3}{3} - 2t = \frac{2}{3}(x^2 - 2)\sqrt{x^2+1}$$

$$\int_0^{\sqrt{3}} \frac{2x^3}{\sqrt{x^2+1}} dx = \left. \frac{2}{3}(x^2 - 2)\sqrt{x^2+1} \right|_0^{\sqrt{3}} = \frac{8}{3}$$

El cambio que hemos hecho es el siguiente:

$$t = \sqrt{x^2+1} \implies x^2 = t^2 - 1 \quad y \quad dt = \frac{x}{\sqrt{x^2+1}} dx$$

4.

$$\int \frac{x^3 - 2x^2}{x^2 - 2x + 1} dx = \int \left(x - \frac{x}{(x-1)^2} \right) dx$$

Hacemos la descomposición en fracciones simples:

$$\frac{x}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} = \frac{A(x-1) + B}{(x-1)^2}$$

$$x = A(x-1) + B$$

Si $x = 1 \implies B = 1$

Si $x = 0 \implies A = B = 1$

$$\int \frac{x^3 - 2x^2}{\sqrt{x^2 - 2x + 1}} dx = \int x dx - \int \frac{1}{x-1} dx - \int \frac{1}{(x-1)^2} dx =$$

$$= \frac{x^2}{2} - \ln|x-1| + \frac{1}{x-1} + C$$

5.

$$\int \frac{1}{x^2 - (a+1)x + a} dx = \int \frac{1}{(x-a)(a-1)} dx$$

Hacemos la descomposición en fracciones simples:

$$\frac{1}{(x-a)(a-1)} = \frac{A}{x-a} + \frac{B}{x-1} = \frac{A(x-1) + B(x-a)}{(x-a)(a-1)}$$

$$1 = A(x-1) + B(x-a)$$

Si $x = 1 \implies B = 1/(1-a)$

Si $x = a \implies A = 1/(a-1)$

$$\int \frac{1}{x^2 - (a+1)x + a} dx = \frac{1}{a-1} \left(\int \frac{1}{x-a} dx - \int \frac{1}{x-1} dx \right) =$$

$$= \frac{1}{a-1} (\ln|x-a| - \ln|x-1|) + C = \frac{1}{a-1} \ln \left| \frac{x-a}{x-1} \right| + C$$

Calcular los siguientes límites:

1. $\lim_{x \rightarrow 0} \frac{(e^x - 1)^2}{e^{x^2} - 1}$ (Extremadura Junio 2008)
2. Calcular el valor de m para que $\lim_{x \rightarrow 0} \frac{mx^2 - 1 + \cos x}{\sin(x^2)} = 0$ (Galicia Junio 2008)
3. $\lim_{x \rightarrow +\infty} (e^x - x^2)$ (Madrid Junio 2008)
4. $\lim_{x \rightarrow +\infty} \frac{4^x + 5^x}{3^x + 6^x}$ (Madrid Junio 2008)
5. $\lim_{x \rightarrow +\infty} \frac{4^x + 5^x}{3^x + 6^x}$ (Madrid Junio 2008)
6. $\lim_{x \rightarrow +\infty} \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+2} - \sqrt{x-2}}$ (Navarra Junio 2008)
7. $\lim_{x \rightarrow \pi/2} \frac{\ln \sqrt{1 - \cos x}}{\ln(1 - \cos x)}$ (Navarra Junio 2008)
8. Calcular el valor de $\alpha \in \mathbb{R}$ para el cual:

$$\lim_{n \rightarrow +\infty} \left(\frac{n^2 - 2n + 1}{n^2 + n - 2} \right)^{\frac{\alpha n^3 + 1}{n^2 - 1}} = 1$$

(Navarra Junio 2008)

Soluciones:

1.

$$\lim_{x \rightarrow 0} \frac{(e^x - 1)^2}{e^{x^2} - 1} = \left[\frac{0}{0} \right] = \lim_{x \rightarrow 0} \frac{2e^x(e^x - 1)}{2xe^{x^2}} = \lim_{x \rightarrow 0} \frac{e^x - 1}{xe^x} = \left[\frac{0}{0} \right] =$$

$$\lim_{x \rightarrow 0} \frac{e^x}{e^x - xe^x} = 1$$

2.

$$\lim_{x \rightarrow 0} \frac{mx^2 - 1 + \cos x}{\sin(x^2)} = \left[\frac{0}{0} \right] = \lim_{x \rightarrow 0} \frac{2mx - \sin x}{2x \cos(x^2)} = \left[\frac{0}{0} \right] =$$

$$\lim_{x \rightarrow 0} \frac{2m - \cos x}{2 \cos(x^2) - 4x^2 \sin(x^2)} = \frac{2m - 1}{2} = 0 \implies m = \frac{1}{2}$$

3.

$$\lim_{x \rightarrow +\infty} (e^x - x^2) = \lim_{x \rightarrow +\infty} e^x \left(1 - \frac{x^2}{e^x}\right) = \lim_{x \rightarrow +\infty} e^x = \infty$$

ya que:

$$\lim_{x \rightarrow +\infty} \frac{x^2}{e^x} = \left[\frac{\infty}{\infty}\right] = \lim_{x \rightarrow +\infty} \frac{2x}{e^x} = \left[\frac{\infty}{\infty}\right] = \lim_{x \rightarrow +\infty} \frac{2}{e^x} = 0$$

4.

$$\lim_{x \rightarrow +\infty} \frac{4^x + 5^x}{3^x + 6^x} = \lim_{x \rightarrow +\infty} \frac{5^x \left(\left(\frac{4}{5}\right)^x + 1\right)}{6^x \left(\left(\frac{3}{6}\right)^x + 1\right)} = \lim_{x \rightarrow +\infty} \left(\frac{5}{6}\right)^x \frac{\left(\frac{4}{5}\right)^x + 1}{\left(\frac{3}{6}\right)^x + 1} = 0$$

5.

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+2} - \sqrt{x-2}} &= \\ \lim_{x \rightarrow +\infty} \frac{(\sqrt{x+1} - \sqrt{x-1})(\sqrt{x+2} + \sqrt{x-2})(\sqrt{x+1} + \sqrt{x-1})}{(\sqrt{x+2} - \sqrt{x-2})(\sqrt{x+2} + \sqrt{x-2})(\sqrt{x+1} + \sqrt{x-1})} &= \\ \lim_{x \rightarrow +\infty} \frac{[(x+1) - (x-1)](\sqrt{x+2} + \sqrt{x-2})}{[(x+2) - (x-2)](\sqrt{x+1} + \sqrt{x-1})} &= \\ \lim_{x \rightarrow +\infty} \frac{2(\sqrt{x+2} + \sqrt{x-2})}{4(\sqrt{x+1} + \sqrt{x-1})} &= \frac{1}{2} \end{aligned}$$

6.

$$\lim_{x \rightarrow \pi/2} \frac{\ln \sqrt{1 - \cos x}}{\ln(1 - \cos x)} = \lim_{x \rightarrow \pi/2} \frac{\frac{1}{2} \ln(1 - \cos x)}{\ln(1 - \cos x)} = \frac{1}{2}$$

7.

$$\begin{aligned} \lim_{n \rightarrow +\infty} \left(\frac{n^2 - 2n + 1}{n^2 + n - 2} \right)^{\frac{\alpha n^3 + 1}{n^2 - 1}} &= [1^\infty] = e^\lambda \\ \lambda &= \lim_{n \rightarrow +\infty} \frac{\alpha n^3 + 1}{n^2 - 1} \left(\frac{n^2 - 2n + 1}{n^2 + n - 2} - 1 \right) = \lim_{n \rightarrow +\infty} \frac{3\alpha n^3 + 3}{-n^3 - 2n^2 + n + 2} = -3\alpha \end{aligned}$$

Luego:

$$\lim_{n \rightarrow +\infty} \left(\frac{n^2 - 2n + 1}{n^2 + n - 2} \right)^{\frac{\alpha n^3 + 1}{n^2 - 1}} = e^{-3\alpha} = 1 \implies -3\alpha = 0 \implies \alpha = 0$$