

Problema 1 Calcular los siguientes límites:

a) $\lim_{x \rightarrow 0} \frac{\ln(\cos 2x)}{x^2}$ (Castilla León 2006)

b) Determinar los valores a y b para los que

$$\lim_{x \rightarrow 0} \frac{ax^2 + bx + 1 - \cos x}{\sin x^2} = 1$$

(Castilla León 2006)

c) $\lim_{x \rightarrow 0} \frac{1 + x - e^x}{\sin^2 x}$ (Extremadura 2006)

d) Si $f(x) = \frac{(x+1)^2}{e^x}$ calcular $\lim_{x \rightarrow \infty} f(x)$ y $\lim_{x \rightarrow -\infty} f(x)$.

(Islas Baleares 2006)

e) Calcular si existen los siguientes límites:

- $\lim_{x \rightarrow 0} (\sin x)^{\tan x}$
- $\lim_{x \rightarrow 0} \frac{\sin x}{|x|}$
- $\lim_{x \rightarrow a} \frac{\sqrt{x} - \sqrt{a}}{x - a}$ con $(x > 0)$

(La Rioja 2006)

f) $\lim_{x \rightarrow 0} \frac{\sin x}{7x^2}$ (Zaragoza)

g) Si $a_n = \frac{2n}{n+1}$ calcular $\lim_{n \rightarrow \infty} n^2(a_{n+1} - a_n)$ (Madrid 2006)

Solución:

a) $\lim_{x \rightarrow 0} \frac{\ln(\cos 2x)}{x^2} = \left[\frac{0}{0} \right] = \lim_{x \rightarrow 0} \frac{\frac{-2 \sin 2x}{\cos 2x}}{2x} = \lim_{x \rightarrow 0} \frac{-\tan 2x}{x} = \left[\frac{0}{0} \right] =$
 $\lim_{x \rightarrow 0} \frac{-2}{\cos^2 x} = -2$

b) $\lim_{x \rightarrow 0} \frac{ax^2 + bx + 1 - \cos x}{\sin x^2} = \left[\frac{0}{0} \right] = \lim_{x \rightarrow 0} \frac{2ax + b + \sin x}{2x \cos x^2} = \left[\frac{0}{0} \right] =$
 $(\text{con } b = 0) = \lim_{x \rightarrow 0} \frac{2a + \cos x}{2 \cos x^2 - 4x^2 \sin x^2} = \frac{2a + 1}{2} = 1 \implies a = \frac{1}{2}$

$$\text{c) } \lim_{x \rightarrow 0} \frac{1+x-e^x}{\sin^2 x} = \left[\frac{0}{0} \right] = \lim_{x \rightarrow 0} \frac{1-e^x}{2 \cos x \sin x} = \lim_{x \rightarrow 0} \frac{1-e^x}{\sin 2x} = \left[\frac{0}{0} \right] = \lim_{x \rightarrow 0} \frac{-e^x}{2 \cos 2x} = -\frac{1}{2}$$

$$\text{d) } \bullet \lim_{x \rightarrow \infty} \frac{(x+1)^2}{e^x} = \left[\frac{\infty}{\infty} \right] = \lim_{x \rightarrow \infty} \frac{2(x+1)}{e^x} = \left[\frac{\infty}{\infty} \right] = \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0$$

$$\bullet \lim_{x \rightarrow -\infty} \frac{(x+1)^2}{e^x} = [\infty \cdot \infty] = \infty$$

$$\text{e) } \bullet \lim_{x \rightarrow 0} (\sin x)^{\tan x} = \lambda \implies \lim_{x \rightarrow 0} \tan x \ln \sin x = \ln \lambda$$

$$\lim_{x \rightarrow 0} \tan x \ln \sin x = \lim_{x \rightarrow 0} \frac{\ln \sin x}{\frac{1}{\tan x}} = \left[\frac{-\infty}{\infty} \right] = \lim_{x \rightarrow 0} \frac{\frac{\cos}{\sin x}}{\frac{-1/\cos^2 x}{\tan^2 x}} = \lim_{x \rightarrow 0} (-\cos x \tan x) = 0 \implies \ln \lambda = 0 \implies \lambda = 1$$

$$\bullet \frac{\sin x}{|x|} = \begin{cases} -\frac{\sin x}{x} & \text{si } x < 0 \\ \frac{\sin x}{x} & \text{si } x \geq 0 \end{cases}$$

$$\lim_{x \rightarrow 0^-} \frac{\sin x}{x} = \lim_{x \rightarrow 0^-} -\frac{\sin x}{x} = \left[\frac{0}{0} \right] = \lim_{x \rightarrow 0^-} -\frac{\cos x}{1} = -1$$

$$\lim_{x \rightarrow 0^+} \frac{\sin x}{x} = \lim_{x \rightarrow 0^+} \frac{\sin x}{x} = \left[\frac{0}{0} \right] = \lim_{x \rightarrow 0^+} \frac{\cos x}{1} = 1$$

Como $\lim_{x \rightarrow 0^-} \frac{\sin x}{x} \neq \lim_{x \rightarrow 0^+} \frac{\sin x}{x}$ concluimos con que el límite no existe.

$$\bullet \lim_{x \rightarrow a} \frac{\sqrt{x} - \sqrt{a}}{x - a} = \left[\frac{0}{0} \right] = \lim_{x \rightarrow a} \frac{\frac{1}{2\sqrt{x}}}{1} = \frac{1}{2\sqrt{a}}$$

$$\text{f) } \lim_{x \rightarrow 0} \frac{\sin x}{7x^2} = \left[\frac{0}{0} \right] = \lim_{x \rightarrow 0} \frac{\cos x}{14x} = \left[\frac{1}{0} \right]$$

$$\lim_{x \rightarrow 0^-} \frac{\cos x}{14x} = \left[\frac{1}{0^-} \right] = -\infty$$

$$\lim_{x \rightarrow 0^+} \frac{\cos x}{14x} = \left[\frac{1}{0^+} \right] = \infty$$

Como $\lim_{x \rightarrow 0^-} \frac{\cos x}{14x} \neq \lim_{x \rightarrow 0^+} \frac{\cos x}{14x}$ concluimos con que el límite no existe.

g)

$$a_n = \frac{2n}{n+1} \implies a_{n+1} = \frac{2(n+1)}{n+2} \implies n^2(a_n - a_{n+1}) = \frac{n^2}{n^2 + 3n + 2}$$

$$\lim_{x \rightarrow \infty} \frac{n^2}{n^2 + 3n + 2} = 1$$